WRINKLING OF AN ELASTIC SHEET FLOATING ON A LIQUID SPHERE

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Abstract. A thin circular elastic sheet floating on a drop-like liquid substrate gets deformed due to incompatibility between the curved substrate and the planar sheet. We adopt a variational viewpoint by minimizing the non-convex membrane energy plus a higher-order convex bending energy. Being interested in thin sheets, we expand the minimum of the energy in terms of a small thickness h, and identify the first two terms of this expansion. The leading order term comes from a minimization of a family of one-dimensional "relaxed" problems, while for the next-order term we only identify its scaling law. This generalizes the previous work [P. Bella and R.V. Kohn. *Wrikling of a thin circular sheet bonded to a spherical substrate*, Philos. Trans. Roy. Soc. A, 375(2017). https://doi.org/10.1098/rsta.2016.0157] to the physically relevant case of a liquid substrate.

1. Introduction

We consider a circular elastic sheet floating on a liquid spherical substrate. Assuming the center of the sheet being the north pole, the sheet tries to wrap the spherical cap, which is not possible without inducing compressive stresses due to the positive curvature of the sphere (substrate). The sheet being thin, rather than by compression, these stresses are relaxed by small-scale wrinkling out of plane. In physics, this type of problems are very popular as prototypical examples of instabilities, and often serve as testing grounds for newly developed theories [21]. Mathematically, it belongs to a large class of problems, where the microstructure can be explained via minimization of non-convex energies – sometimes referred to as energy-driven pattern formation [20].

The model of a circular sheet floating on a deformable ball was, from the mathematical point of view, previously studied by Kohn and the first author [6], under the assumption that the underlying compliant substrate is made from an elastic material. While mathematically convenient (see Section 2), from the physical point of view the more favorable situation is with a liquid substrate. The goal of this paper is to generalize the previous result [6] to allow for the case of a liquid substrate.

There are many areas of material science, where the occurrence and properties of microstructure were successfully explained based on the energy minimization: magnetics [17, 25], type-I superconductors [12] (see also [27, 28, 29] for recent developments in type-II superconductivity), shape-memory alloys [2], diblock copolymers [13], to name just a few. In all of them, one minimizes a non-convex energy regularized with a higher-order term. The non-convex part of the

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energy favors oscillations, while the higher-order part limits the speed of them. The regularizing term having small prefactor (in the present case played by the non-dimensionalized thickness of the sheet), the weaker regularization allows for rapid oscillations and gives rise to a microstructure.

Before we proceed to the actual statements, let us put our result into perspective. Over the last decade, there was a vast activity in studying deformations of thin elastic sheets, in particular their ability to deal with compressive stresses. To allow for experiments with extremely thin sheets with thickness as small as several tens of nanometers, one usually place them on a liquid bath – either straight or curved as in the 6case of a droplet [21, 26]. A flat elastic sheet lying on a curved substrate induces necessary stresses [16, 18, 19], the compressive ones usually being relaxed by the out-of-plane oscillations. These deformations might be quite large, hence requiring new physical arguments based on the nonlinear stability theory [32]. Recently, new physical arguments to take advantage of localized delamination to release the incompatibility in some case were analyzed [11, 14].

On the mathematical side, analysis of elastic sheets via scaling laws goes back as far as e.g. [8]. Motivated by the physical experiments [15], the first author and Kohn [4] identified the scaling law for the first order term in the energy expansions in the case of an annular thin sheet stretched in the radial direction (see also [1, 7] for identification of the optimal prefactor and corresponding Γ-convergence result, respectively). While in [8, 9, 10] the prescribed metric was biaxial compression, the situation for compression in only one direction (the second direction being free or in tension) is completely different [3, 5] – not only in terms of the scaling law (power law $h^{4/3}$ vs linear), but also in terms of very different construction of the upper bound as well as arguments for the lower bound, and also in terms of the spatial energy distribution (localized near the boundary vs relatively uniform distribution). While we have mentioned some, there are many other examples of prestrained elastic sheets, for example in biological applications [22].

In other instances, the incompatibility is caused by the prescribed boundary conditions instead of a non-Euclidean metric. In a series of works, Olbermann [23, 24] considered a d-cone problem – prescribing conical boundary conditions, one asks how the conical singularity gets smoothed out in order to have finite bending energy. The problem is very non-trivial, in particular at the level of the lower bound, since it is not even clear whether the modification of the cone should be of local nature.

The previously mentioned results were all dealing with analysis of the next-order term in the energy (excess energy), which describes the properties of microstructure (wrinkling), since the leading order term is in the flat case well understood via relaxation. Nevertheless, in the curved situation already the leading-order term is not well understood, see recent works by Tobasco [30, 31].

2. Preliminaries and main results

2.1. **Heuristic arguments.** Before we discuss the form of the energy, let us try to heuristically understand the situation. Assuming first for simplicity that the sheet does not stretch in the radial direction, circles at distance r from the pole (center of the sheet) should then map to shorter circles. Said differently, distance along the sphere from the pole to a circle of radius r is longer than r - simply due to the curvature of the sphere. If the sheet is thin enough, it prefers to oscillate out-of-plane rather than to compress, which leads to the formation of wrinkles.

Assuming small deformations of the sheet, we model the elastic energy of the sheet of thickness h using a simplified "Föppl-von Kárman" energy - consisting of a membrane part (measuring deviation of the midplane deformation from being an isometry) plus a bending part which penalizes the curvature. The first term has prefactor h, which comes from the volume, while the bending is multiplied by a factor h^3. We do not assume any connection between the sheet and the substrate, hence the sheet is allowed to flow freely on it. Since we do not allow for cavities (delamination) between the sheet and the substrate, the only energetic contributions from the substrate are related to the gravity (pulling up or pushing down the fluid caused by the out-of-plane displacement of the sheet). Normalizing the whole energy by the volume (i.e., dividing by h), the substrate prefactor scales like h^{-1}. In comparison, in [6] the first author and Kohn considered an elastic substrate, in which case the prefactor was of order h^{-2} – the exponent coming from the scaling of the $H^{1/2}$-norm of the out-of-plane displacement, such norm being proxy for the elastic energy of substrate (see discussion in [6, Section 2]). For a more general treatment, in the present paper we will allow for any scaling $h^{-\beta}$ with $\beta \in (0, 2]$ instead of just $\beta = 2$.

We consider a thin elastic circular sheet sitting on top of a solid elastic ball. To conform with the results in the physics community and since we assume to be in the regime of relatively small (at least in-plane) deformations, as announced, instead of the nonlinear elasticity we consider a Föppl-von Kárman energy, with a simplified role of the in-plane displacement. Both the sheet as well as the substrate possess a radial symmetry, so it is convenient to rewrite the energy using polar coordinates.

We denote by $u = (u_r, u_\theta)$ and ξ the in-plane (the radial and hoop part) and out-plane *displacements*, respectively. Farther, we denote by r_0 and R respectively the radii of the circular sheet and the liquid ball underneath. The Föppl-von Kárman energy in the radial variables then has the form

$$E_h(u,\xi) = \int_0^{r_0}\int_0^{2\pi} \Bigg[\left|\partial_r u_r + \frac{(\partial_r\xi)^2}{2}\right|^2 + \left|\frac{\partial_\theta u_\theta}{r} + \frac{u_r}{r} + \frac{(\partial_\theta \xi)^2}{2r^2}\right|^2 + 2\left|\frac{\partial_\theta u_r}{2r} + \frac{\partial_r u_\theta}{2} + \frac{\partial_r\xi\partial_\theta\xi}{2r}\right|^2$$

$$h^2\left(\frac{|\partial_{\theta\theta}\xi|^2}{r^4} + |\partial_{rr}\xi|^2 + \frac{2|\partial_{\theta r}\xi|^2}{r^2}\right) + \frac{\alpha_s}{h^\beta}\left|\xi + \frac{r^2}{2R}\right|^2 \Bigg] \,\mathrm{d}\theta\; r\,\mathrm{d}r.$$

Here, the first three terms represent the membrane energy, in the rectangular coordinates having the form $|e(u) + \frac{1}{2}\nabla\xi \otimes \nabla\xi|^2$ with $e(u) := \frac{1}{2}(\nabla u + \nabla^T u)$ denoting the symmetric part of the gradient, and the fourth term with prefactor h^2 stands for the bending energy. While these four terms constitute the usual Föppl-von Kárman energy, the last term is related to the cost of deformation of the substrate. To be consistent with the previous approximation, instead of describing the spherical shape of the substrate by $R - R(1 - (r/R)^2)^{1/2}$, we replace it by the term $\frac{r^2}{2R}$, which is valid under the assumption $r \ll R$.

The parameter $\beta \in (0, 2]$ in the prefactor $\alpha_s h^{-\beta}$ plays a decisive role in the problem. To draw a contrast, in [6] only the "critical case" $\beta = 2$ was considered as a proxy for the case of an elastic substrate (see [6] for the details). In that case the cost of wrinkling (coming basically from both the bending and the substrate part of the energy) contributes to the leading term of the energy – naively, bending has two derivatives and prefactor h^2, membrane part has one derivative with no h, and substrate term has no derivative and prefactor h^{-2} – so, scalingwise, they match well. In particular, the "leading order" (relaxed) energy has a non-trivial form involving a term related to wrinkling (i.e. the cost of wrinkling enters this part of the energy). Hence the limiting shape (while factoring out small-scale oscillations due to wrinkling) will be deformed in the radial direction, i.e., the radial $_{rr}$ part of the stress will not vanish. In particular, in contrast to a "weaker" substrate as considered here, the limiting shape is strained in the radial direction – see the form of the minimizer $\mathbf{u}_h^0$ of the relaxed/limiting problem in (2.3) for the case $\beta = 2$, which is naturally h-independent.

In contrast, in the case $\beta \in (0,2)$, the case of liquid substrate ($\beta = 1$) being a special case, the minimum of the "leading order" energy converges to 0 (see Remark 2.1), and the h-dependent minimizers $\mathbf{u}_h^0$ converge to a trivial shape, isometric in the radial direction. Nevertheless, we manage to analyze the behavior of the energy (see Theorem 2.1 below), which can be seen as a difference between the minimum of the full energy and the 1-dimensional h-dependent proxy functional F_h^0, which describes the leading order energy.

In addition, in the present paper we also improve the understanding of the critical case $\beta = 2$, by replacing the upper bound $Ch\exp(C|\log(h)|^{1/2}\log(-\log(h)))$ from [6] to a much better $Ch|\log(h)|^C$.

2.2. **The energy.** Besides the out-of-plane displacement ξ, it will be often convenient to consider the relative out-of-plane displacement

$$w(r,\theta) := \xi(r,\theta) + \frac{r^2}{2R}.$$

Using polar coordinates, it is convenient to introduce averaging $\bar{f} := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$ for $f \in L^1(0,2\pi)$.

Using simple manipulations (see beginning of [6, Section 3], in particular [6, (3.5-3.7)]), we can rewrite the energy E_h

$$E_h(u,w) = \int_0^{r_0} \left(\left| \bar{u}_r' + \frac{1}{2}\left(\frac{r}{R} - \bar{w}'\right)^2 + B(r) \right|^2 + W_r\left(\frac{\bar{u}_r}{r}, w\right) + h^2 \left| \bar{w}'' - \frac{1}{R} \right|^2 + \frac{\alpha_s}{h^\beta}|\bar{w}|^2 \right) r\,\mathrm{d}r + R_h(u,\xi), \tag{2.1}$$

where $B(r) := \overline{|\partial_r(\bar{w} - w)|^2}$,

$$W_r(\eta, w) := \left| \eta + \frac{\overline{|\partial_\theta w|^2}}{2r^2} \right|^2 + h^2 \frac{\overline{|\partial_{\theta\theta} w|^2}}{r^4} + \frac{\alpha_s}{h^\beta}\overline{|w - \bar{w}|^2},$$

and the non-negative remainder

$$R_h(u,\xi) = \int_0^{r_0} \fint_0^{2\pi} \left(\left| \partial_r(u_r - \bar{u}_r) + \frac{(\partial_r \xi)^2}{2} - \frac{\overline{(\partial_r \xi)^2}}{2} \right|^2 + \left| \frac{\partial_\theta u_\theta}{r} + \frac{u_r - \bar{u}_r}{r} + \frac{(\partial_\theta w)^2}{2r^2} - \frac{\overline{(\partial_\theta w)^2}}{2r^2} \right|^2 + \frac{1}{2}\left| \frac{\partial_\theta u_r}{r} + r\partial_r\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\partial_r\xi\partial_\theta\xi \right|^2 + h^2\left|\partial_{rr} w\right|^2 + \frac{2h^2}{r^2}\left|\partial_{\theta r}\xi\right|^2 \right) \mathrm{d}\theta\, r\,\mathrm{d}r.$$

Though the function $W_r(\eta, w)$ depends also on the thickness parameter h, to keep the notation simple we will avoid writing this dependence.

2.3. **Energetic cost of elastic deformations of a circle – analysis of W_r.** An essential computation in the analysis is estimating the energetic cost of elastic deformations (with the out-of-plane displacement ξ) of a circle being fit into either smaller or larger space (modeled by $-\eta$).

A slightly simpler situation, that in turn will represent pretty well ours, is that of deforming a horizontally positioned straight elastic rope of length ℓ to make it occupy a different amount of space. Letting one of its end points fixed and moving the other by an amount η, the minimal energetic cost (using as before the Föppl-von Kárman approximation) of such elastic deformation is given by

$$W_{\mathrm{rel}}(\eta) := \min_{u(0)=0,\ u(\ell)=\eta} \int_0^\ell \left| u' + \frac{1}{2}\vartheta'^2 \right|^2 + h^2|\vartheta''|^2 + \alpha_s h^{-\beta}|\vartheta|^2,$$

where u and ϑ represent respectively the horizontal and vertical deformations.

Observe that given any function ϑ, we can explicitly find a function w_ϑ for which the energy is minimal among all possible horizontal deformations. The Euler–Lagrange equation for the minimizer w_ϑ reads $u_\vartheta'' + \frac{1}{2}(\vartheta'^2)' = 0$, that is $u_\vartheta' + \frac{1}{2}\vartheta'^2 = C$ for some constant C. Integrating this equality, we find $u_\vartheta(\ell) + \frac{1}{2}\int \vartheta'^2 = C$. Replacing $u_\vartheta(\ell) = \eta$, we deduce that our original minimization problem is equivalent to

$$\min \left(\eta + \frac{1}{2}\int_0^\ell \vartheta'^2 \right)^2 + \int_0^\ell h^2|\vartheta''|^2 + \alpha_s h^{-\beta}|\vartheta|^2.$$

We immediately see that if $\eta \geq 0$, it is optimal to deform the rope only horizontally, that is $\vartheta \equiv 0$. In this case, the optimal energetic cost is equal to η^2. This is also the case if η is slightly negative, similarly as in the case of buckling of rods. For more negative η the wrinkling instability occurs, where the out-of-plane deformation is used as a way to waste some arclength.

In order to find the optimal energetic cost in the case $\eta < 0$, let us observe that for periodic ϑ, using integration by parts and the Cauchy-Schwarz inequality we get

$$\int_0^\ell h^2|\vartheta''|^2 + \alpha_s h^{-\beta}|\vartheta|^2 \geq 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \int_0^\ell |\vartheta'|^2.$$

This inequality being sharp, we see that the minimization problem can be reduced to

$$\min_{M\geq 0} \left(\eta + \frac{1}{2}M \right)^2 + 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} M$$

with $M = \int_0^\ell \vartheta'^2$. Choosing the optimal $M = \max\big(-2\eta - 4\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}, 0\big)$, we get

$$W_{\mathrm{rel}}(\eta) = \begin{cases} \eta^2 & \text{if } \eta \geq -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \\ -4\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} (\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} + \eta) & \text{if } \eta \leq -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}. \end{cases} \tag{2.2}$$

2.4. **Effective functional.** The leading order behavior of $\min_{u,\xi} E_h$, for h sufficiently small, is obtained by minimizing the *effective functional*

$$F_h^0(\mathbf{u}) := \int_0^{r_0} \left[\left(\mathbf{u}' + \frac{r^2}{2R^2} \right)^2 + W_{\mathrm{rel}}\left(\frac{\mathbf{u}}{r} \right) \right] r\,\mathrm{d}r,$$

where $\mathbf{u}$ is a radial real-valued function constrained by the boundary condition $\mathbf{u}(0) = 0$. This functional reflects a competition between the elastic energy of radial tension and the elastic cost of deforming circles derived in the previous subsection. We consider F_h^0 for two reasons: 1) we can explicitly write its minimizer (see (2.3)) and 2) it captures quite well the behavior of E_h (Theorem 2.1).

The functional F_h^0 is strictly convex, and the unique solution of the corresponding Euler-Lagrange equation has the form

$$\mathbf{u}_h^0(r) = \begin{cases} -\dfrac{3}{16}\dfrac{r^3}{R^2} + \left(2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\left(\dfrac{r_0}{r_h} - 1\right) + \dfrac{1}{16}\dfrac{r_h^2}{R^2}\right) r & \text{for } r \in [0, r_h] \\ -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r - \dfrac{1}{6}\left(\dfrac{r^3 - r_h^3}{R^2}\right) + 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 \log \dfrac{r}{r_h} & \text{for } r \in [r_h, r_0], \end{cases} \tag{2.3}$$

where $r_h := \left(16\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 R^2\right)^{\frac{1}{3}}$. Observe that in the first region $(0, r_h)$ we have $\mathbf{u}_h^0(r)/r \geq -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}$, i.e., this is exactly the region where W_{rel} is quadratic, whereas $W_{\text{rel}}(\mathbf{u}_h^0(r)/r)$ is affine in (r_h, r_0).

In the above computation we implicitly assumed that $r_h < r_0$, a condition which is satisfied for any $h \leq 1$ and any $\beta \in (0, 2]$ whenever $\alpha_s < 2^{-8}\left(\frac{r_0}{R}\right)^4$ if $\beta = 2$ or h is sufficiently small if $\beta \in (0, 2)$. It is convenient to introduce the stress

$$\sigma_h^0 := (\mathbf{u}_h^0)' + \frac{r^2}{2R^2},$$

notation that will be used throughout the rest of the paper.

As a technical commodity, hereafter we will assume that $r_h \leq \frac{1}{3} r_0$, a condition which holds if $\alpha_s \leq 3^{-6} 2^{-8} r_0^4 R^{-4}$ in the case $\beta = 2$ or if h is sufficiently small when $\beta \in (0, 2)$. The advantage of this assumption is just cosmetic, as can be seen in Section 3.3. A minor modification of the proof contained there allows to treat the general case $r_h < r_0$, with the disadvantage that one needs to keep track of the number $d := r_0 - r_h$ throughout the proof.

Remark 2.1. *Using* (2.3)*, one can compute*

$$F_h^0(\mathbf{u}_h^0) = C h^{\frac{2-\beta}{2}} + o(h^{\frac{2-\beta}{2}}),$$

where C is a constant that depends on α_s, r_0, and R.

2.5. **Main results.** Our main result identifies the scaling law for the minimum of the excess energy (difference between minimum of E_h and the leading order term $F_h^0(\mathbf{u}_h^0)$), for which we naturally need to specify the space X over which we will minimize. For X we choose the largest reasonable space for which the energy makes sense:

$$X := \{(u, \xi) : (0, r_0) \times [0, 2\pi] \mapsto \mathbb{R}^2 \times \mathbb{R}, u \in W^{1,2}, \xi \in W^{2,2}\}.$$

Theorem 2.1. *Let $\beta \in (0, 2]$ and $\alpha_s \in (0, 3^{-6} 2^{-8} r_0^4 R^{-4})$. There exist positive constants h_0,c_0, c_1, c_3, c_4 depending on α_s, r_0, and R (with h_0 also depending on β if $\beta \leq \frac{2}{3}$), such that for any $0 < h < h_0$, we have*

$$c_0 h^{\frac{6-\beta}{4}} \leq \inf_{(u,\xi) \in X} E_h(u, \xi) - F_h^0(\mathbf{u}_h^0) \leq \begin{cases} c_1 |\log h|^{c_2} h^{\frac{6-\beta}{4}} & \text{if } \frac{2}{3} \leq \beta \leq 2 \\ c_3 |\log h|^{c_4} h^{\frac{2+\beta}{2}} & \text{if } 0 \leq \beta \leq \frac{2}{3}. \end{cases}$$

Remark 2.2. *In the "critical case" $\beta = 2$, this result improves the upper bound contained in [6]. Since the lower bound does not change, we will only consider the case $\beta = 2$ in the proof of the upper bound part of this theorem in Section 4.*

Theorem 2.2. *Letting $\sigma(\bar{u}_r, \bar{w}) := \bar{u}_r' + \frac{1}{2}\left(\frac{r}{R} - \bar{w}'\right)^2$ and defining*

$$X_+ := \{(u, \xi) \in X \mid \sigma(\bar{u}_r, \bar{w}) \geq 0\}, \tag{2.4}$$

if $0 \le \beta \le \frac{2}{3}$*, then there exist positive constants* h_0, c_3, c_4 *depending on* α_s*,* r_0*,* R *(with* h_0 *also depending on* β *if* $\beta \le \frac{2}{3}$*), such that for any* $0 < h < h_0$*, we have*

$$c_0 h^{\frac{4}{3}} \le \inf_{(u,\xi)\in X_+} E_h(u,\xi) - F_h^0(\mathbf{u}_h^0) \le c_3 |\log h|^{c_4} h^{\frac{2+\beta}{2}}.$$

Remark 2.3. *The condition* $\sigma(\bar{u}_r, \bar{w}) \ge 0$ *seems physically natural, since it expresses that the sheet is stretched in the radial direction. Nevertheless, we were not able to show its validity, and had to assume it instead.*

Remark 2.4. *In Theorem 2.2 it is not necessary to assume any condition on the parameters* α_s*,* r_0*, and* R*, since, in the regime* $\beta \le \frac{2}{3}$*, for any* h *sufficiently small (independently of* β*) we have* $r_h < \frac{1}{3} r_0$*.*

3. Lower bound

3.1. **The functional F_h.** We will consider another "effective" functional lying between the full energy E_h and the previously defined functional F_h^0. While F_h^0 was only depending on the averaged displacement in the radial direction, the functional F_h also considers averages of the out-of-plane displacement – but still being relatively accessible since it is defined on functions of r only. As we will see in the next proposition, F_h and F_h^0 are still close to each other (minimizers and the energies of these two functionals differ very little) while F_h is at the same time much better (at least for analytic reasons) approximation of the full energy E_h.

Motivated by (2.1), we define F_h as a functional of $(\mathbf{u}, \mathbf{w})$

$$F_h(\mathbf{u}, \mathbf{w}) := \int_0^{r_0} \left[\left(\mathbf{u}' + \frac{1}{2} \left(\frac{r}{R} - \mathbf{w}' \right)^2 \right)^2 + W_{\mathrm{rel}} \left(\frac{\mathbf{u}}{r} \right) + h^2 \left| \mathbf{w}'' - \frac{1}{R} \right|^2 + \alpha_s h^{-\beta} |\mathbf{w}|^2 \right] r \, \mathrm{d}r.$$

Observe that $F_h^0(\mathbf{u}) = F_h(\mathbf{u}, 0) - \frac{h^2 r_0^2}{2R^2}$ for any admissible $\mathbf{u}$, i.e. F_h is like a generalization of the functional F_h^0. Moreover, at the level of minimizers, the term $\alpha_s h^{-\beta} |\mathbf{w}|^2$ forces $\mathbf{w}$ to be small in L^2, though still allowing for possibly rapid oscillations. For these oscillations to be useful we would nevertheless need $\mathbf{w}' \sim r/R$ in some sense to allow $\mathbf{u}' \sim 0$, and subsequently $\mathbf{u} \sim 0$, which in turn would force large L^2-norm of $\mathbf{w}$. The following result makes these observations quantitative.

Proposition 3.1. *The functional* F_h *has a unique minimizer* $(\mathbf{u}_h, \mathbf{w}_h)$ *and there exists* $C = C(\alpha_s, r_0, R)$ *such that*

$$|F_h^0(\mathbf{u}_h^0) - F_h(\mathbf{u}_h, \mathbf{w}_h)| \le C h^2. \tag{3.1}$$

Moreover,

$$\|\mathbf{u}_h^0 - \mathbf{u}_h\|_{L^\infty\left(\left(\left(\frac{1}{4} r_h, r_0\right)\right)\right)} \le C h^{\frac{2+\beta}{4}} |\log h|^{1/2} \tag{3.2}$$

and the stress

$$\sigma_h(r) := \mathbf{u}_h' + \frac{1}{2} \left(\frac{r}{R} - \mathbf{w}_h' \right)^2$$

satisfies, for any $r \in (0, r_0)$*,*

$$\sigma_h(r) \ge 0. \tag{3.3}$$

In addition, for h *sufficiently small (independently of* β*) we have*

$$\sigma_h(r) = 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \left(\frac{r_0}{r} - 1 \right) \quad \text{for any } r \in \left[\frac{1}{2} r_0, r_0 \right]. \tag{3.4}$$

Proof. We first show that F_h has a unique minimizer. For this, we consider the modified functional

$$\widetilde{F}_h(\mathbf{u},\mathbf{w}) \;:=\; \int_0^{r_0}\left[\left(\mathbf{u}'+\frac{1}{2}\left(\frac{r}{R}-\mathbf{w}'\right)^2\right)_+^2 \;+\;\alpha_s h^{-\beta}|\mathbf{w}|^2\;+\;h^2\left|\mathbf{w}''-\frac{1}{R}\right|^2\;+\;W_{\mathrm{rel}}\left(\frac{\mathbf{u}}{r}\right)\right] r\,\mathrm{d}r.$$

Since W_{rel} is a convex function and the function $(s,t)\mapsto(s+t^2/2)^2$ is convex in the region $s+t^2/2\ge 0$, we see that $\widetilde{F}_h$ is convex, and strictly convex whenever $\mathbf{u}'+\frac{1}{2}\left(\frac{r}{R}-\mathbf{w}'\right)^2\ge 0$. Thanks to this convexity, the direct method of the calculus of variations yields the existence of a minimizer $(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)$. Using $\widetilde{F}_h(\mathbf{u},\mathbf{w})\le F_h(\mathbf{u},\mathbf{w})$ we observe that $(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)$ also minimizes F_h provided that we show $\widetilde{F}_h(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)=F_h(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)$. Denoting $\widetilde{\sigma}:=(\widetilde{\mathbf{u}}_h'+\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2)_+$, the minimizer $(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)$ has to satisfy the Euler–Lagrange equation $(-2\widetilde{\sigma}r)'+W_{\mathrm{rel}}'(\widetilde{\mathbf{u}}_h/r)=0$. To show that $\widetilde{\mathbf{u}}_h'+\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2\ge 0$ let us assume the contrary, i.e., there exists a point $r^*\in(0,r_0)$ with $\widetilde{\mathbf{u}}_h'(r^*)+\frac{1}{2}\big(\frac{r^*}{R}-\widetilde{\mathbf{w}}_h'(r^*)\big)^2<0$. Since both $\widetilde{\mathbf{u}}_h'$ and $\widetilde{\mathbf{w}}_h'$ are twice weakly differentiable, the latter being true since $\widetilde{\mathbf{w}}_h\in W^{2,2}(0,r_0)$ and the former coming from the Euler–Lagrange equation, we have that $\widetilde{\mathbf{u}}_h'+\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2<0$ in a neighborhood $\mathcal{N}$ of r^*. Since $\widetilde{\sigma}=0$ in $\mathcal{N}$, it follows from the Euler–Lagrange equation that $W_{\mathrm{rel}}'\big(\frac{\widetilde{\mathbf{u}}_h(r)}{r}\big)=(2r\widetilde{\sigma})'=0$ there. Since $W_{\mathrm{rel}}'(t)=0$ if and only if $t=0$, it follows that $\widetilde{\mathbf{u}}_h=0$ in $\mathcal{N}$, in particular $\widetilde{\mathbf{u}}_h'+\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2=\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2\ge 0$ in $\mathcal{N}$, a contradiction.

Having shown $\widetilde{\mathbf{u}}_h'+\frac{1}{2}\big(\frac{r}{R}-\widetilde{\mathbf{w}}_h'\big)^2\ge 0$ implies two facts: combined with strict convexity of $\widetilde{F}_h$ in the region $\widetilde{\sigma}\ge 0$, we have uniqueness of $(\widetilde{\mathbf{u}}_h,\widetilde{\mathbf{w}}_h)$ as the minimizer of $\widetilde{F}_h$, and in turn also as the unique minimizer $(\mathbf{u}_h,\mathbf{w}_h)$ of F_h, which in addition also satisfies (3.3).

The $(\mathbf{u}_h,\mathbf{w}_h)$ being the unique minimizer of F_h, let us now show the closeness of $\mathbf{u}_h$ to $\mathbf{u}_h^0$. By minimality of $\mathbf{u}_h^0$ and $(\mathbf{u}_h,\mathbf{w}_h)$ for F_h^0 and F_h respectively, we have that

$$DF_h(\mathbf{u}_h,\mathbf{w}_h)=0 \quad\text{and}\quad DF_h^0(\mathbf{u}_h^0)=D_{\mathbf{u}}F_h(\mathbf{u}_h^0,0)=0, \tag{3.5}$$

where DF_h (resp. DF_h^0) denotes the Fréchet derivative of F_h (resp. F_h^0) and $D_{\mathbf{u}}F_h$ the Gateaux derivative of F_h in the direction $(\mathbf{u},0)$.

In addition, for any φ smooth we compute

$$D_{\mathbf{w}}F_h(\mathbf{u}_h^0,0)[\varphi]=-2\int_0^{r_0}\sigma_h^0\frac{r}{R}\varphi' r\,\mathrm{d}r.$$

Assuming φ is compactly supported, integrating by parts and using the Euler–Lagrange equation $(2\sigma_h^0 r)'=W_{\mathrm{rel}}'\big(\frac{\mathbf{u}_h^0(r)}{r}\big)$, which comes from variation of F_h in $\mathbf{u}$, we deduce that

$$D_{\mathbf{w}}F_h(\mathbf{u}_h^0,0)[\varphi]=\frac{1}{R}\int_0^{r_0}\left[W_{\mathrm{rel}}'\left(\frac{\mathbf{u}_h^0}{r}\right)+2\sigma_h^0\right]\varphi r\,\mathrm{d}r.$$

A direct computation using (2.3) shows that

$$\int_0^{r_0}\left|W_{\mathrm{rel}}'\left(\frac{\mathbf{u}_h^0}{r}\right)+2\sigma_h^0\right|^2 r\,\mathrm{d}r\le Ch^{2-\beta},$$

where throughout the proof C denotes a constant that depends only on r_0, R, and α_s, that may change from line to line. Using the Cauchy–Schwartz inequality, we then find

$$|D_{\mathbf{w}}F_h(\mathbf{u}_h^0,0)[\varphi]|\le Ch^{\frac{2-\beta}{2}}\left(\int_0^{r_0}|\varphi|^2 r\,\mathrm{d}r\right)^{\frac{1}{2}}. \tag{3.6}$$

Observing that the gradients of F_h at $(\mathbf{u}_h^0, 0)$ and $(\mathbf{u}_h, \mathbf{w}_h)$ are small (or even vanish), uniform convexity of F_h would imply closeness of these points - something one can show using Taylor expansion to the second order. Though F_h is not convex, we still can proceed in this way.

We define $f(x,y) := \left(x + \frac{1}{2}y^2\right)^2$ and for any $x, y, a, b \in \mathbb{R}$ get

$$
\begin{aligned}
(3.7)\quad & f(x,y) - f(a,b) - Df(a,b)[x-a, y-b] \\
&= \left(\left(x + \frac{1}{2}y^2\right) - \left(a + \frac{1}{2}b^2\right)\right)^2 + \left(a + \frac{1}{2}b^2\right)(y-b)^2.
\end{aligned}
$$

Indeed, observe that the first term on the r.h.s comes from the Taylor expansion of $g(t) = t^2$ in the form $t^2 - s^2 - 2s(t-s) = (t-s)^2$, whereas the second term on the r.h.s is the difference between Df and the gradient of g.

Using the above identify for f and Taylor's theorem with remainder in the integral form, we find

$$
\begin{aligned}
(3.8)\quad & F_h(\mathbf{u}_h^0, 0) - F_h(\mathbf{u}_h, \mathbf{w}_h) - DF_h(\mathbf{u}_h, \mathbf{w}_h)[\mathbf{u}_h^0 - \mathbf{u}_h, -\mathbf{w}_h] \\
&= \int_0^{r_0} \Bigg[(\sigma_h^0 - \sigma_h)^2 + \sigma_h (\mathbf{w}_h')^2 + \alpha_s h^{-\beta} \mathbf{w}_h^2 + h^2 (\mathbf{w}_h'')^2 \\
&\qquad + \int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r} W_{\mathrm{rel}}''(\eta) \left(\frac{\mathbf{u}_h^0}{r} - \eta\right) \mathrm{d}\eta \Bigg] r \,\mathrm{d}r
\end{aligned}
$$

and

$$
\begin{aligned}
& F_h(\mathbf{u}_h, \mathbf{w}_h) - F_h(\mathbf{u}_h^0, 0) - DF_h(\mathbf{u}_h^0, 0)[\mathbf{u}_h - \mathbf{u}_h^0, \mathbf{w}_h] \\
&= \int_0^{r_0} \Bigg[(\sigma_h^0 - \sigma_h)^2 + \sigma_h^0 (\mathbf{w}_h')^2 + \alpha_s h^{-\beta} \mathbf{w}_h^2 + h^2 (\mathbf{w}_h'')^2 \\
&\qquad + \int_{\mathbf{u}_h^0/r}^{\mathbf{u}_h/r} W_{\mathrm{rel}}''(\eta) \left(\frac{\mathbf{u}_h}{r} - \eta\right) \mathrm{d}\eta \Bigg] r \,\mathrm{d}r.
\end{aligned}
$$

Summing these two relations, and using (3.5), yields

$$
\begin{aligned}
(3.9)\quad -D_{\mathbf{w}} F_h(\mathbf{u}_h^0, 0)[\mathbf{w}_h] &= \int_0^{r_0} \Bigg[2(\sigma_h^0 - \sigma_h)^2 + (\sigma_h^0 + \sigma_h)(\mathbf{w}_h')^2 + 2\alpha_s h^{-\beta} \mathbf{w}_h^2 \\
&\qquad + 2h^2 (\mathbf{w}_h'')^2 + \frac{\mathbf{u}_h^0 - \mathbf{u}_h}{r} \int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r} W_{\mathrm{rel}}''(\eta) \mathrm{d}\eta \Bigg] r \,\mathrm{d}r.
\end{aligned}
$$

Since $W_{\mathrm{rel}}'' \geq 0$ and both σ_h^0 and σ_h are non-negative, all the terms on the right-hand side in the previous relation are non-negative. Thus, using (3.6), we deduce that

$$
h^{-\beta} \int_0^{r_0} \mathbf{w}_h^2 r \,\mathrm{d}r \leq C h^{\frac{2-\beta}{2}} \left(\int_0^{r_0} \mathbf{w}_h^2 r \,\mathrm{d}r \right)^{\frac{1}{2}},
$$

that is

$$
\left(\int_0^{r_0} \mathbf{w}_h^2 r \,\mathrm{d}r \right)^{\frac{1}{2}} \leq C h^{\frac{2+\beta}{2}}.
$$

This, together with (3.6), gives

$$
|D_{\mathbf{w}} F_h(\mathbf{u}_h^0, 0)[\mathbf{w}_h]| \leq C h^2.
$$

Inserting in (3.9), we find

$$\int_0^{r_0}\Big[2(\sigma_h^0-\sigma_h)^2+(\sigma_h^0+\sigma_h)(\mathbf{w}_h')^2+2\alpha_s h^{-\beta}\mathbf{w}_h^2 + 2h^2(\mathbf{w}_h'')^2+\frac{\mathbf{u}_h^0-\mathbf{u}_h}{r}\int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r}W_{\mathrm{rel}}''(\eta)\mathrm{d}\eta\Big]r\,\mathrm{d}r\le Ch^2. \tag{3.10}$$

Since the left-hand side bounds from above the right-hand side in (3.8), using (3.5), we deduce that

$$0\le F_h(\mathbf{u}_h^0,0)-F_h(\mathbf{u}_h,\mathbf{w}_h)\le Ch^2,$$

the lower bound coming from the optimality of $(\mathbf{u}_h,\mathbf{w}_h)$. This combined with the fact $F_h(\mathbf{u}_h^0,0)=F_h^0(\mathbf{u}_h^0)+h^2\frac{r_0^2}{2R^2}$ yields (3.1).

Let us now provide some estimates on $\mathbf{u}_h$ and $\mathbf{w}_h$. From (3.10) together with non-negativity of all the terms, we immediately deduce that

$$\int_0^{r_0}\mathbf{w}_h^2 r\,\mathrm{d}r\le Ch^{2+\beta}\quad\text{and}\quad\int_0^{r_0}(\mathbf{w}_h'')^2r\,\mathrm{d}r\le C.$$

By the Gagliardo–Nirenberg interpolation inequality, we find

$$\int_0^{r_0}\mathbf{w}_h'^4 r\,\mathrm{d}r\le C\left(\int_0^{r_0}\mathbf{w}_h^2r\,\mathrm{d}r\right)^{\frac12}\left(\int_0^{r_0}\mathbf{w}_h''^2r\,\mathrm{d}r\right)^{\frac32}+\frac{C}{r_0^6}\left(\int_0^{r_0}\mathbf{w}_h^2r\,\mathrm{d}r\right)^2\le Ch^{\frac{2+\beta}{2}}$$

and

$$\int_0^{r_0}\mathbf{w}_h'^2 r\,\mathrm{d}r\le C\left(\int_0^{r_0}\mathbf{w}_h^2r\,\mathrm{d}r\right)^{\frac12}\left(\int_0^{r_0}\mathbf{w}_h''^2r\,\mathrm{d}r\right)^{\frac12}+\frac{C}{r_0^2}\int_0^{r_0}\mathbf{w}_h^2r\,\mathrm{d}r\le Ch^{\frac{2+\beta}{2}}.$$

From (3.10), we also have $\int_0^{r_0}(\sigma_h^0-\sigma_h)^2r\,\mathrm{d}r\le Ch^2$, which combined with the previous two inequalities and $\sigma_h^0-\sigma_h=((\mathbf{u}_h^0)'-\mathbf{u}_h')+\frac{r}{R}\mathbf{w}_h'-\frac{r^2}{2R^2}(\mathbf{w}_h')^2$ yields

$$\int_0^{r_0}|(\mathbf{u}_h^0)'-\mathbf{u}_h'|^2r\,\mathrm{d}r\le Ch^{\frac{2+\beta}{2}}. \tag{3.11}$$

We will now estimate $\int_0^{r_0}|\mathbf{u}_h^0-\mathbf{u}_h|^2r\,\mathrm{d}r$ using the last term in (3.10). Heuristically, recalling $W_{\mathrm{rel}}''=2\chi_{(-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}},\infty)}$, that term in (3.10) will control $|\mathbf{u}_h^0-\mathbf{u}_h|^2$ provided $\mathbf{u}_h(r)\ge-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}r$. If $\mathbf{u}_h(r)<-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}r$, that term controls at least $\mathbf{u}_h^0-\mathbf{u}_h$ provided the "prefactor" $\int_{-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}}^{\mathbf{u}_h^0/r}2$ is not too small. To make this rigorous, let us consider the interval

$$A_h:=\left(\frac14 r_h,\frac12 r_h\right),$$

and define the sets

$$A_h^1:=\left\{r\in A_h\;\middle|\;\frac{\mathbf{u}_h(r)}{r}\ge-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\right\}\quad\text{and}\quad A_h^2:=\left\{r\in A_h\;\middle|\;\frac{\mathbf{u}_h(r)}{r}<-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\right\}.$$

Since $W_{\mathrm{rel}}''(\eta)=2$ for any $\eta\ge-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}$, and recalling that $\frac{\mathbf{u}_h^0(r)}{r}\ge-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}$ for $r\in(0,r_h)$, from (3.10) we deduce that

$$\int_{A_h^1}\frac{|\mathbf{u}_h^0-\mathbf{u}_h|^2}{r}\mathrm{d}r=\frac12\int_{A_h^1}\frac{\mathbf{u}_h^0-\mathbf{u}_h}{r}\left(\int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r}W_{\mathrm{rel}}''(\eta)\mathrm{d}\eta\right)r\,\mathrm{d}r\le Ch^2. \tag{3.12}$$

On the other hand, observe that for any $r \in A_h^2$, we have

$$\int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r} W''_{\rm rel}(\eta){\rm d}\eta = \int_{-2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}}^{\mathbf{u}_h^0/r} 2\,{\rm d}\eta = 2\left(\frac{\mathbf{u}_h^0}{r} + 2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right) \geq Cr_h^2,$$

the last estimate coming from explicit computation using (2.3): since $2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}} \geq 0$ (both in the above relation as well as in definition (2.3)), we see that $\frac{\mathbf{u}_h^0}{r} \geq \frac{1}{16R^2}(r_h^2 - 3r^2) \geq \frac{1}{16R^2}\frac{r_h^2}{4}$, the last step coming from $r \leq r_h/2$.

Combining this with (3.10), and using that $\mathbf{u}_h(r) \leq -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}} r$ implies $\mathbf{u}_h \leq \mathbf{u}_h^0$ in A_h^2, we get

$$\int_{A_h^2} |\mathbf{u}_h^0 - \mathbf{u}_h|\,{\rm d}r = \int_{A_h^2} \frac{\mathbf{u}_h^0 - \mathbf{u}_h}{r} r\,{\rm d}r \leq C\int_{A_h^2} \frac{\mathbf{u}_h^0 - \mathbf{u}_h}{r}\left(\frac{1}{r_h^2}\int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r} W''_{\rm rel}(\eta){\rm d}\eta\right) r\,{\rm d}r$$
$$\leq C\int_0^{r_0} \frac{\mathbf{u}_h^0 - \mathbf{u}_h}{r}\left(\frac{1}{r_h^2}\int_{\mathbf{u}_h/r}^{\mathbf{u}_h^0/r} W''_{\rm rel}(\eta){\rm d}\eta\right) r\,{\rm d}r, \leq C\frac{h^2}{r_h^2}, \tag{3.13}$$

where the last but one step follows from the non-negativity of the integrand.

From (3.12) and (3.13), using the Cauchy–Schwartz inequality, we deduce that

$$\int_{A_h} |\mathbf{u}_h^0 - \mathbf{u}_h|\,{\rm d}r \leq Cr_h\left(\int_{A_h^1} \frac{|\mathbf{u}_h^0 - \mathbf{u}_h|^2}{r}\,{\rm d}r\right)^{\frac12} + \int_{A_h^2} (\mathbf{u}_h^0 - \mathbf{u}_h)\,{\rm d}r \leq Cr_h h + C\frac{h^2}{r_h^2},$$

which implies

$$\frac{1}{|A_h|}\int_{A_h} |\mathbf{u}_h^0 - \mathbf{u}_h|\,{\rm d}r \leq Ch. \tag{3.14}$$

From (3.11) and Hölder's inequality follows

$$\int_{\frac14 r_h}^{r_0} |\mathbf{u}_h^{0\,\prime} - \mathbf{u}_h'|\,{\rm d}r \leq \left(\int_{\frac14 r_h}^{r_0} |\mathbf{u}_h^{0\,\prime} - \mathbf{u}_h'|^2 r\,{\rm d}r\right)^{1/2}\left(\int_{\frac14 r_h}^{r_0} \frac{{\rm d}r}{r}\right)^{1/2} \leq Ch^{\frac{2+\beta}{4}}|\log(Cr_h)|^{1/2}.$$

This combined with (3.14) and the (1-dimensional) Poincaré–Wirtinger inequality

$$\int_{\frac14 r_h}^{r_0} \left|(\mathbf{u}_h^0 - \mathbf{u}_h) - \frac{1}{|A_h|}\int_{A_h} (\mathbf{u}_h^0 - \mathbf{u}_h)\,{\rm d}r'\right| {\rm d}r \leq C\int_{\frac14 r_h}^{r_0} |\mathbf{u}_h^{0\,\prime} - \mathbf{u}_h'|\,{\rm d}r,$$

where C does not depend on h, we deduce that

$$\int_{\frac14 r_h}^{r_0} |\mathbf{u}_h^0 - \mathbf{u}_h|\,{\rm d}r \leq \frac{r_0 - \frac14 r_h}{|A_h|}\int_{A_h} |\mathbf{u}_h^0 - \mathbf{u}_h|\,{\rm d}r + C\int_{\frac14 r_h}^{r_0} |\mathbf{u}_h^{0\,\prime} - \mathbf{u}_h'|\,{\rm d}r \leq Ch^{\frac{2+\beta}{4}}|\log h|^{1/2}.$$

In addition, by Sobolev embedding of $W^{1,1}$ into L^∞ in $\mathbb{R}$, we deduce that

$$\|\mathbf{u}_h^0(r) - \mathbf{u}_h(r)\|_{L^\infty((\frac14 r_h, r_0))} \leq Ch^{\frac{2+\beta}{4}}|\log h|^{1/2}.$$

Explicit computations on $\mathbf{u}_h^0(r)$ for $r > r_h$ combined with the previous estimate, allow to deduce that $\frac{\mathbf{u}_h(r)}{r} \leq -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}$ for any $r \in I = \left[\frac12 r_0, r_0\right]$, provided h is sufficiently small (independently of β) – see Step 1 in the first proof presented in Section 3.3. Using the fact that $\mathbf{u}_h$ satisfies the Euler–Lagrange equation

$$(2\sigma_h r)' = W'_{\rm rel}\left(\frac{\mathbf{u}_h}{r}\right) = -4\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}$$

in I with $\sigma_h(r_0) = 0$, a straight forward computation gives (3.4). This conclude s the proof of the proposition. □

The following proposition provides estimates on several non-negative quantities in terms of the excess energy $\inf E_h - F_h^0(\mathbf{u}_h^0)$, hence it is the crucial step towards the lower bound.

Proposition 3.2. *There exists $C = C(\alpha_s, r_0, R)$ such that*

(3.15)
$$\inf_{(u,\xi)\in X} \int_0^{r_0} \left(\sigma_h B + W_r\left(\frac{\bar u_r}{r},\xi\right) - W_{\rm rel}\left(\frac{\bar u_r}{r}\right) + \left(\left(\frac{\mathbf{u}_h}{r} \vee -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right) - \left(\frac{\bar u_r}{r} \vee -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right)\right)^2\right) r\,\mathrm{d}r$$
$$+ h^{\frac{2-\beta}{2}}\left(\int_0^{r_0} (\bar u_r' - \mathbf{u}_h')_+\, r\,\mathrm{d}r\right)^2 + R_h(u,\xi)$$
$$\le C\left(\inf_{(u,\xi)\in X} E_h(u,\xi) - F_h^0(\mathbf{u}_h^0)\right) + O(h^2),$$

where hereafter $a \vee b := \max\{a,b\}$. Besides, recalling (2.4)*, we have*

(3.16)
$$\inf_{(u,\xi)\in X_+} \int_0^{r_0} \left(\sigma_h B + W_r\left(\frac{\bar u_r}{r},\xi\right) - W_{\rm rel}\left(\frac{\bar u_r}{r}\right) + \left(\left(\frac{\mathbf{u}_h}{r} \vee -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right) - \left(\frac{\bar u_r}{r} \vee -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right)\right)^2\right) r\,\mathrm{d}r$$
$$+ \int_0^{r_0} B^2 r\,\mathrm{d}r + h^{\frac{2-\beta}{2}}\left(\int_0^{r_0} (\bar u_r' - \mathbf{u}_h')_+\, r\,\mathrm{d}r\right)^2 + R_h(u,\xi)$$
$$\le C\left(\inf_{(u,\xi)\in X_+} E_h(u,\xi) - F_h^0(\mathbf{u}_h^0)\right) + O(h^2).$$

Proof. Let us begin by estimating $E_h(u,\xi) - F_h(\mathbf{u}_h,\mathbf{w}_h)$. Rewriting (2.1), we have

(3.17)
$$E_h(u,\xi) - F_h(\mathbf{u}_h,\mathbf{w}_h) =$$
$$\tilde F_h(\bar u_r,\bar w,B) - F_h(\mathbf{u}_h,\mathbf{w}_h) + \int_0^{r_0}\left(W_r\left(\frac{\bar u_r}{r},\xi\right) - W_{\rm rel}\left(\frac{\bar u_r}{r}\right)\right) r\,\mathrm{d}r + R_h(u,\xi),$$

where

$$\tilde F_h(\bar u_r,\bar w,B) := \int_0^{r_0}\left(|\sigma(\bar u_r,\bar w) + B|^2 + W_{\rm rel}\left(\frac{\bar u_r}{r}\right) + h^2\left|\bar w'' - \frac1R\right|^2 + \frac{\alpha_s}{h^\beta}|\bar w|^2\right) r\,\mathrm{d}r,$$
$$F_h(\mathbf{u},\mathbf{w}) = \int_0^{r_0}\left(|\sigma_h|^2 + W_{\rm rel}\left(\frac{\mathbf{u}}{r}\right) + h^2\left|\mathbf{w}'' - \frac1R\right|^2 + \frac{\alpha_s}{h^\beta}|\mathbf{w}|^2\right) r\,\mathrm{d}r.$$

We recall that both $W_r - W_{\rm rel}$ as well as R_h on the r.h.s. of (3.17) are non-negative.

Using (3.7), $DF_h(\mathbf{u}_h,\mathbf{w}_h) = 0$, and Taylor's theorem with remainder in the integral form, we find

(3.18)
$$\varepsilon := \tilde F_h(\bar u_r,\bar w,B) - F_h(\mathbf{u}_h,\mathbf{w}_h) =$$
$$\int_0^{r_0}\Bigg[(\sigma(\bar u_r,\bar w) - \sigma_h + B)^2 + \sigma_h(\bar w' - \mathbf{w}_h')^2 + \alpha_s h^{-\beta}(\bar w - \mathbf{w}_h)^2$$
$$+ h^2(\bar w'' - \mathbf{w}_h'')^2 + 2\sigma_h B + \int_{\mathbf{u}_h/r}^{\bar u_r/r} W_{\rm rel}''(\eta)\left(\frac{\bar u_r}{r} - \eta\right)\mathrm{d}\eta\Bigg] r\,\mathrm{d}r.$$

Let us observe that all the terms on the right-hand side are non-negative, in particular also thanks to $\sigma_h \ge 0$ (see (3.3)).

In order to get the estimate on $\int_0^{r_0}(\bar{u}_r' - \mathbf{u}_h')_+ \, r \, \mathrm{d}r$, we observe that

$$\int_0^{r_0} (\bar{w} - \mathbf{w}_h)^2 r \, \mathrm{d}r \le C\varepsilon h^{\beta} \quad \text{and} \quad \int_0^{r_0} (\bar{w}'' - \mathbf{w}_h'')^2 r \, \mathrm{d}r \le C\varepsilon h^{-2}.$$

By interpolation inequality, we deduce that

$$\int_0^{r_0} (\bar{w}' - \mathbf{w}_h')^2 r \, \mathrm{d}r \le C\varepsilon h^{\frac{\beta-2}{2}},$$

which implies $\int_0^{r_0} |\bar{w}' - \mathbf{w}_h'| r \, \mathrm{d}r \le C\varepsilon^{\frac{1}{2}} h^{\frac{\beta-2}{4}}$. Since $B \ge 0$, it implies

$$\begin{aligned}
\int_0^{r_0} (\bar{u}_r' - \mathbf{u}_h')_+ \, r \, \mathrm{d}r &\le \int_0^{r_0} |\bar{u}_r' - \mathbf{u}_h' + B| \, r \, \mathrm{d}r \\
&\le C \int_0^{r_0} \left(|\sigma(\bar{u}_r, \bar{w}) - \sigma_h + B| + \left| \left(\frac{r}{R} - \bar{w}'\right)^2 - \left(\frac{r}{R} - \mathbf{w}_h'\right)^2 \right| \right) r \, \mathrm{d}r \\
&\le C(\varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} h^{\frac{\beta-2}{4}}) \le C\varepsilon^{\frac{1}{2}} h^{\frac{\beta-2}{4}},
\end{aligned}$$

the last estimate coming from the upper bound on ε.

Finally, to estimate $\left(\left(\frac{\mathbf{u}_h}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\right) - \left(\frac{\bar{u}_r}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\right)\right)^2$, observe that $W''(\eta) = 2$ in the region $\eta \ge -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}$, and so $\left(\left(\frac{\mathbf{u}_h}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\right) - \left(\frac{\bar{u}_r}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\right)\right)^2$ is directly bounded by the term $\int_{\mathbf{u}_h/r}^{\bar{u}_r/r} W_{\mathrm{rel}}''(\eta) \left(\frac{\bar{u}_r}{r} - \eta\right) \mathrm{d}\eta$ contained in (3.18).

In the case $(u, \xi) \in X_+$

$$\tilde{F}_h(\bar{u}_r, \bar{w}, B) = F_h(\bar{u}_r, \bar{w}) + \int_0^{r_0} (2\sigma(\bar{u}_r, \bar{w})B + B^2) \, r \, \mathrm{d}r \ge F_h(\mathbf{u}_h, \mathbf{w}_h) + \int_0^{r_0} B^2 r dr.$$

Hence

$$\varepsilon \ge \int_0^{r_0} B^2 \, r \, \mathrm{d}r.$$

This combined with the previous result gives (3.16). □

3.2. **Energetic costs of changing the wavenumber.** The following lemma is crucial to identify the next-order term in the expansion of the energy. At the heuristic level, this next order consists of two main contribution: penalization of not having optimal number of wrinkles (wavelength) as assumed in the derivation of (2.2), and the cost of changing this wavenumber as a function of r. The assumption on η_i there means that we will see some wrinkling, since otherwise the above consideration would not hold.

Lemma 3.1. *Let $\delta \ge 0$ and $\rho_0, \rho_1 \in (0, r_0)$ with $\rho_0 < \rho_1$ be such that*

$$\eta_i := \frac{\bar{u}_r(\rho_i)}{\rho_i} \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \delta$$

for $i = 0, 1$. Then, letting $\lambda = \rho_1 - \rho_0$, we have

$$\begin{aligned}
(3.19) \quad \sum_{i=0}^{1} (W_{\rho_i}(\eta_i, \xi) - W_{\mathrm{rel}}(\eta_i)) + \alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \fint_{\rho_0}^{\rho_1} B(r) \, \mathrm{d}r& \\
\ge \frac{\delta}{2} \min\left(\frac{\delta}{2}, \left(\frac{6\rho_1^4}{\rho_0^2} \frac{1}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \lambda^2} + \frac{(\rho_0 + \rho_1)^2}{4\rho_0^2} \frac{\lambda^2}{h^2} \right)^{-1} \right)&.
\end{aligned}$$

Proof. We denote $\{a_k(r)\}_{k\in\mathbb{Z}}$ the Fourier coefficients of $w(r,\cdot)$. Then, using Plancherel, we have

$$B(r) = \sum_{k\neq 0} (a_k'(r))^2.$$

Moreover, letting

$$A(r) := \frac{1}{2r^2}\sum_{k\neq 0} a_k(r)^2 k^2,$$

directly using the definition of W_r, we obtain

$$\begin{aligned} W_r(\eta,\xi) &= |\eta + A(r)|^2 + \frac{1}{r^2}\sum_{k\neq 0} a_k(r)^2k^2\left[\left(\frac{h|k|}{r} - \frac{\alpha_s^{\frac12} r}{h^{\frac{\beta}{2}}|k|}\right)^2 + 2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right] \\ &= |\eta + A(r)|^2 + 4\alpha_s^{\frac12} h^{\frac{2-\beta}{2}} A(r) + \frac{1}{r^2}\sum_{k\neq 0} a_k(r)^2k^2\left(\frac{h|k|}{r} - \frac{\alpha_s^{\frac12} r}{h^{\frac{\beta}{2}}|k|}\right)^2. \end{aligned}$$

For the rest of the proof we assume $a_0(r) = 0$ for all $r\in[\rho_0,\rho_1]$, since it simplifies the notation without changing the problem.

Recalling (2.2), for $\eta \le -2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}$, we have

$$W_r(\eta,\xi) - W_{\mathrm{rel}}(\eta) = \left|A(r) + \left(\eta + 2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}}\right)\right|^2 + \frac{1}{r^2}\sum_{k} a_k(r)^2k^2\left(\frac{h|k|}{r} - \frac{\alpha_s^{\frac12} r}{h^{\frac{\beta}{2}}|k|}\right)^2. \tag{3.20}$$

Since by hypothesis $\eta_i + 2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}} \le -\delta$, assuming $A(\rho_i) \le \frac{\delta}{2}$ either for $i=0$ or $i=1$, we would get

$$|A(r) + (\eta_i + 2\alpha_s^{\frac12} h^{\frac{2-\beta}{2}})| \ge \frac{\delta}{2},$$

and the result would immediately follow.

Hence it remains to analyze the case $A(\rho_0) \ge \frac{\delta}{2}, A(\rho_1) \ge \frac{\delta}{2}$. Let $k_i > 0$ be the optimal wavenumber defined via

$$\frac{hk_i}{\rho_i} - \frac{\alpha_s^{\frac12}\rho_i}{h^{\frac{\beta}{2}} k_i} = 0,$$

that is $k_i = \dfrac{\alpha_s^{\frac14}\rho_i}{h^{\frac{\beta+2}{4}}}$ for $i=0,1$. Let us also define $K := \dfrac{k_1-k_0}{2} = \dfrac{\alpha_s^{\frac14}\lambda}{2h^{\frac{\beta+2}{4}}}$.

On one hand, letting $A_K(\rho_0) = \frac{1}{2\rho_0^2}\sum_{||k|-k_0|\geq K} a_k(\rho_0)^2k^2$, we have

$$\begin{aligned}
W_r(\eta_0,\xi) - W_{\rm rel}(\eta_0) &\geq \frac{1}{\rho_0^2}\sum_{||k|-k_0|\geq K} a_k(\rho_0)^2k^2\left(\frac{h|k|}{\rho_0} - \frac{\alpha_s^{\frac12}\rho_0}{h^{\frac{\beta}{2}}|k|}\right)^2\\
&= \frac{1}{\rho_0^2}\sum_{||k|-k_0|\geq K} a_k(\rho_0)^2k^2\frac{h^2}{\rho_0^2}\left(\frac{|k|^2-k_0^2}{|k|}\right)^2\\
&\geq \frac{1}{\rho_0^2}\sum_{||k|-k_0|\geq K} a_k(\rho_0)^2k^2\frac{h^2}{\rho_0^2}||k|-k_0|^2\\
&\geq 2\frac{1}{2\rho_0^2}\frac{h^2}{\rho_0^2}K^2\sum_{||k|-k_0|\geq K} a_k(\rho_0)^2k^2\\
&\geq 2K^2\frac{h^2}{\rho_0^2}A_K(\rho_0) = \frac12\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\frac{\lambda^2}{\rho_0^2}A_K(\rho_0),
\end{aligned}$$

that is

$$A_K(\rho_0) \leq \frac{2\rho_0^2}{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\lambda^2}(W_{\rho_0}(\eta_0,\xi) - W_{\rm rel}(\eta_0)). \tag{3.21}$$

In words, $A_K(\rho_0)$ denotes the energy of wrinkles with sub-optimal wavelengths.

On the other hand, since

$$\begin{aligned}
a_k(\rho_0)^2 &\leq 2a_k(\rho_1)^2 + 2|a_k(\rho_1) - a_k(\rho_0)|^2\\
&\leq 2a_k(\rho_1)^2 + 2\left(\int_{\rho_0}^{\rho_1} a_k'(r)\,\mathrm{d}r\right)^2\\
&\leq 2a_k(\rho_1)^2 + 2\lambda\int_{\rho_0}^{\rho_1}(a_k'(r))^2\,\mathrm{d}r,
\end{aligned}$$

we have for the energy of wrinkles with close to optimal wavelengths

$$\begin{aligned}
A(\rho_0) - A_K(\rho_0) &= \frac{1}{2\rho_0^2}\sum_{||k|-k_0|<K} a_k(\rho_0)^2k^2\\
&\leq \frac{\rho_1^2}{\rho_0^2}\frac{1}{\rho_1^2}\sum_{||k|-k_0|<K} a_k(\rho_1)^2k^2 + \lambda\int_{\rho_0}^{\rho_1}\sum_{||k|-k_0|<K}\frac{k^2}{\rho_0^2}(a_k'(r))^2.\,\mathrm{d}r
\end{aligned} \tag{3.22}$$

From the definition $K = \frac{k_1-k_0}{2}$ it follows $\{k : ||k|-k_0| < K\} \subset \{k : ||k|-k_1| \geq K\}$, hence

$$\sum_{||k|-k_0|<K} a_k(\rho_1)^2k^2 \leq \sum_{||k|-k_1|\geq K} a_k(\rho_1)^2k^2 \leq \sum_{||k|-k_1|\geq K} a_k(\rho_1)^2k^2\frac{||k|-k_1|^2}{K^2}.$$

Combining with

$$||k|-k_1|^2 \leq \left(\frac{|k|^2-k_1^2}{|k|}\right)^2 = \frac{\rho_1^2}{h^2}\left(\frac{h|k|}{\rho_1} - \frac{\alpha_s^{\frac12}\rho_1}{h^{\frac{\beta}{2}}|k|}\right)^2,$$

we deduce that

$$\frac{\rho_1^2}{\rho_0^2}\frac{1}{\rho_1^2}\sum_{||k|-k_0|<K} a_k(\rho_1)^2k^2 \le \frac{\rho_1^4}{h^2\rho_0^2K^2}\frac{1}{\rho_1^2}\sum_k a_k(\rho_1)^2k^2\left(\frac{h|k|}{\rho_1}-\frac{\alpha_s^{\frac12}\rho_1}{h^{\frac{\beta}{2}}|k|}\right)^2 \le \frac{4\rho_1^4}{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\rho_0^2\lambda^2}(W_{\rho_1}(\eta_1,\xi)-W_{\mathrm{rel}}(\eta_1)). \tag{3.23}$$

Besides, by observing that, for any k such that $||k|-k_0|\le K$, one has

$$|k| \le \frac{k_0+k_1}{2} = \frac{\alpha_s^{\frac14}(\rho_0+\rho_1)}{2h^{\frac{\beta+2}{4}}}, \tag{3.24}$$

we deduce that

$$\lambda\int_{\rho_0}^{\rho_1}\sum_{||k|-k_0|<K}\frac{k^2}{\rho_0^2}(a_k'(r))^2\,\mathrm{d}r \le \frac{(\rho_0+\rho_1)^2\lambda^2}{4\rho_0^2h^2}\alpha_s^{\frac12}h^{\frac{2-\beta}{2}} \fint_{\rho_0}^{\rho_1} B(r)dr.$$

By plugging in this and (3.23) into (3.22), we are led to

$$A(\rho_0)-A_K(\rho_0) \le \left(\frac{4\rho_1^4}{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\rho_0^2\lambda^2}+\frac{(\rho_0+\rho_1)^2\lambda^2}{4\rho_0^2h^2}\right)\left(W_{\rho_1}(\eta_1,\xi)-W_{\mathrm{rel}}(\eta_1)+\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\fint_{\rho_0}^{\rho_1}B(r)\,\mathrm{d}r\right).$$

Summing this and (3.21), and using the fact that $A(\rho_0)\ge\frac{\delta}{2}$, we deduce that

$$\frac{\delta}{2}\left(\frac{6\rho_1^4}{\rho_0^2}\frac{1}{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\lambda^2}+\frac{(\rho_0+\rho_1)^2\lambda^2}{4\rho_0^2h^2}\right)^{-1} \le \sum_{i=0}^1(W_{\rho_i}(\eta_i,\xi)-W_{\mathrm{rel}}(\eta_i))+\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\fint_{\rho_0}^{\rho_1}B(r)\,\mathrm{d}r.$$

This concludes the proof. □

Remark 3.1. *Provided $\delta>0$ and h is sufficiently small, the minimum on the right-hand side of (3.19) is achieved by matching the terms depending on h and λ, that is choosing $\lambda^4=C_1h^{\frac{2+\beta}{2}}$, which then gives*

$$\sum_{i=0}^1(W_{\rho_i}(\eta_i,\xi)-W_{\mathrm{rel}}(\eta_i))+\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\fint_{\rho_0}^{\rho_1}B(r)\,\mathrm{d}r \ge C_2h^{\frac{6-\beta}{4}}, \tag{3.25}$$

where C_1, C_2 are constants that depend on ρ_0,ρ_1,α_s, and δ.

Lemma 3.2. *Let $\delta\ge0$ and $\rho_0,\rho_1\in(0,r_0)$ with $\rho_0<\rho_1$ be such that*

$$\eta_i := \frac{\bar u_r(\rho_i)}{\rho_i} \le -2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}-\delta$$

for $i=0,1$. Then, letting $\lambda=\rho_1-\rho_0$, we have

$$\sum_{i=0}^1(W_{\rho_i}(\eta_i,\xi)-W_{\mathrm{rel}}(\eta_i))+\fint_{\rho_0}^{\rho_1}B^2(r)\,\mathrm{d}r \ge \frac{\delta}{2}\min\left\{\frac{\delta}{2},\frac{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\lambda^2}{2\rho_0^2},\frac{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\lambda^2\rho_0^2}{2\delta\rho_1^2},\frac{\delta}{8}\left(\frac{8\rho_1^4}{\rho_0^2}\frac{1}{\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\lambda^2}+\frac{\alpha_s(\rho_0+\rho_1)^4}{2\rho_0^4}\frac{\lambda^4}{h^{2+\beta}}\right)^{-1}\right\}. \tag{3.26}$$

Proof. Both the statement and the proof are similar to Lemma 3.1, and we use the same notation. Arguing as in that proof, we are left with the case $A(\rho_i) \geq \frac{\delta}{2}$ for $i = 0, 1$. Once again, we define $k_i = \dfrac{\alpha_s^{\frac{1}{4}} \rho_i}{h^{\frac{\beta+2}{4}}}$ for $i = 0, 1$ and $K := \dfrac{k_1 - k_0}{2} = \dfrac{\alpha_s^{\frac{1}{4}} \lambda}{2h^{\frac{\beta+2}{4}}}$.

Letting $A_K(\rho_0) := \frac{1}{2\rho_0^2} \sum_{||k|-k_0| \geq K} a_k(\rho_0)^2 k^2$, we first consider the case

$$A_K(\rho_0) \geq \frac{A(\rho_0)}{2}.$$

Observe that by (3.20)

$$\begin{aligned} W_{\rho_0}(\eta_0, \xi) - W_{\text{rel}}(\eta_0) &\geq \frac{1}{\rho_0^2} \sum_{||k|-k_0| \geq K} a_k(\rho_0)^2 k^2 \left(\frac{h|k|}{\rho_0} - \frac{\alpha_s^{\frac{1}{2}} \rho_0}{h^{\frac{\beta}{2}} |k|} \right)^2 \\ &= \frac{1}{\rho_0^2} \sum_{||k|-k_0| \geq K} a_k(\rho_0)^2 k^2 \frac{h^2}{\rho_0^2} \left(\frac{|k|^2 - k_0^2}{|k|} \right)^2 \\ &\geq \frac{1}{\rho_0^2} \sum_{||k|-k_0| \geq K} a_k(\rho_0)^2 k^2 \frac{h^2}{\rho_0^2} ||k| - k_0|^2 \\ &\geq 2K^2 \frac{h^2}{\rho_0^2} A_K(\rho_0), \end{aligned}$$

which by $A_K(\rho_0) \geq A(\rho_0)/2 \geq \delta/4$ yields

$$\frac{\delta}{4} \frac{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \lambda^2}{2\rho_0^2} \leq W_{\rho_0}(\eta_0, \xi) - W_{\text{rel}}(\eta_0). \tag{3.27}$$

The remaining case is $A_K(\rho_0) < A(\rho_0)$, meaning

$$A(\rho_0) - A_K(\rho_0) \geq \frac{A(\rho_0)}{2}.$$

Observe that, by Hölder's inequality, we have

$$\begin{aligned} a_k(\rho_0)^2 &\leq 2a_k(\rho_1)^2 + 2|a_k(\rho_1) - a_k(\rho_0)|^2 \\ &\leq 2a_k(\rho_1)^2 + 2 \left(\int_{\rho_0}^{\rho_1} a_k'(r) \, \mathrm{d}r \right)^2 \\ &\leq 2a_k(\rho_1)^2 + 2\lambda^{\frac{3}{2}} \left(\int_{\rho_0}^{\rho_1} (a_k'(r))^4 \, \mathrm{d}r \right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

(3.28)

$$\begin{aligned}\frac{\delta}{4} \le A(\rho_0) - A_K(\rho_0) &= \frac{1}{2\rho_0^2} \sum_{||k|-k_0|<K} a_k(\rho_0)^2 k^2 \\ &\le \frac{\rho_1^2}{\rho_0^2}\frac{1}{\rho_1^2} \sum_{||k|-k_0|<K} a_k(\rho_1)^2 k^2 + 2\lambda^{\frac{3}{2}} \left(\int_{\rho_0}^{\rho_1} \sum_{||k|-k_0|<K} \frac{k^4}{\rho_0^4} (a_k'(r))^4 \, dr \right)^{\frac{1}{2}} \\ &\le \frac{4\rho_1^4}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2} (W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1)) + 2\lambda^{\frac{3}{2}} \left(\int_{\rho_0}^{\rho_1} \sum_{||k|-k_0|<K} \frac{k^4}{\rho_0^4} (a_k'(r))^4 \, \mathrm{d}r \right)^{\frac{1}{2}},\end{aligned}$$

where the last estimate can be shown as in Lemma 3.1 (see (3.23)).

We will now consider two cases. If

$$1 \le \frac{4\rho_1^4}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2} (W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1))$$

then

$$\frac{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2}{4\rho_1^4} \le W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1). \tag{3.29}$$

If not, then by squaring (3.28) and using (3.23), we find

$$\begin{aligned}\frac{\delta^2}{16} &\le 2\left(\frac{4\rho_1^4}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2} (W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1)) \right)^2 + 8\lambda^3 \int_{\rho_0}^{\rho_1} \sum_{||k|-k_0|<K} \frac{k^4}{\rho_0^4} (a_k'(r))^4 \, \mathrm{d}r \\ &\le \frac{8\rho_1^4}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2} (W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1)) + 8\lambda^3 \int_{\rho_0}^{\rho_1} \sum_{||k|-k_0|<K} \frac{k^4}{\rho_0^4} (a_k'(r))^4 \, \mathrm{d}r,\end{aligned}$$

where the last inequality comes from the reverse of (3.29). Then, (3.24) yields

$$\lambda^3 \int_{\rho_0}^{\rho_1} \sum_{||k|-k_0|<K} \frac{k^4}{\rho_0^4} (a_k'(r))^4 \, \mathrm{d}r \le \frac{\alpha_s (\rho_0 + \rho_1)^4 \lambda^4}{16 \rho_0^4 h^{2+\beta}} \fint_{\rho_0}^{\rho_1} B^2(r) \, \mathrm{d}r.$$

From these two relations, we deduce that

$$\frac{\delta^2}{16} \left(\frac{8\rho_1^4}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \rho_0^2 \lambda^2} + \frac{\alpha_s (\rho_0 + \rho_1)^4 \lambda^4}{2\rho_0^4 h^{2+\beta}} \right)^{-1} \le (W_{\rho_1}(\eta_1, \xi) - W_{\mathrm{rel}}(\eta_1)) + \fint_{\rho_0}^{\rho_1} B^2(r) \, \mathrm{d}r.$$

By combining this with (3.27), and (3.29), we find (3.26). □

Remark 3.2. *Provided $\delta > 0$ and h is sufficiently small, the minimum on the right-hand side of* (3.26) *is achieved by matching the terms*

$$\frac{8\rho_1^4}{\rho_0^2} \frac{1}{\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \lambda^2} \sim \frac{\alpha_s (\rho_0 + \rho_1)^4}{2\rho_0^4} \frac{\lambda^4}{h^{2+\beta}},$$

that is $\lambda = C_1 h^{\frac{2+3\beta}{12}}$, which then implies

$$\sum_{i=0}^{1} (W_r(\eta_i, \xi) - W_{\mathrm{rel}}(\eta_i)) + \fint_{\rho_0}^{\rho_1} B^2(r) \, \mathrm{d}r \ge C_2 h^{\frac{4}{3}}. \tag{3.30}$$

Moreover, from this and (3.25)*, we deduce that*

$$\sum_{i=0}^{1}(W_r(\eta_i,\xi)-W_{\rm rel}(\eta_i))+\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}\fint_{\rho_0}^{\rho_1}B(r)\,\mathrm{d}r+\fint_{\rho_0}^{\rho_1}B^2(r)\,\mathrm{d}r$$
$$\geq C\max\{h^{\frac43},h^{\frac{6-\beta}{4}}\}=C\begin{cases} h^{\frac{6-\beta}{4}} & \text{if } \frac23\leq\beta<2 \quad (\text{with } \lambda\sim h^{\frac{2+\beta}{8}}) \\ h^{\frac43} & \text{if } 0\leq\beta\leq\frac23 \quad (\text{with } \lambda\sim h^{\frac{2+3\beta}{12}}).\end{cases}$$

Here, C, C_1, C_2 *are constants that depend on* ρ_0, ρ_1, α_s*, and* δ*.*

3.3. **Proof of the lower bound part of the main results.**

Proof of the lower bound part of Theorem 2.1. Let

$$I_1:=\left[\frac12 r_0,\frac{7}{12}r_0\right]\quad\text{and}\quad I_2:=\left[\frac23 r_0,r_0\right],$$

where by $r_h\leq\frac13 r_0$ both are included in the interval $[r_h,r_0]$.

Step 1. To be able to use Lemma 3.1 and Lemma 3.2, we need to show that for r at least in I_2 the value $\bar u_r(r)/r$ lies well below $-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}$ (i.e. the sheet should wrinkle there, and not only barely).

More precisely, we claim that there exist $h_0,\delta_0>0$ (where only h_0 depends on β in the case $\beta\in\left(0,\frac23\right]$) such that, for any $0<h<h_0$ and for almost every $r\in I_2$,

$$\frac{\bar u_r(r)}{r}\leq -2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}-\delta_0. \tag{3.31}$$

Before giving the argument, let us explain the reasoning. By a direct computation, we know that (3.31) is true if we replace $\bar u_r$ by $\mathbf{u}_h^0$. By (3.2), we know that $\mathbf{u}_h^0$ and $\mathbf{u}_h$ do not differ much, i.e. this also holds for $\mathbf{u}_h$, and it remains to compare $\mathbf{u}_h$ and $\bar u_r$. For that we use (3.15), precisely the control on $\int_0^{r_0}(\bar u_r'-\mathbf{u}_h')_+r\,\mathrm{d}r$, which for convenience we replace by $\int_{\frac13 r_0}^{r_0}(\bar u_r'-\mathbf{u}_h')_+\,\mathrm{d}r$. In particular we have for any interval $[a,b]\subset\left[\frac13 r_0,r_0\right]$ that

$$(\bar u_r-\mathbf{u}_h)(b)-(\bar u_r-\mathbf{u}_h)(a)=\int_a^b(\bar u_r'-\mathbf{u}_h')(r)\,\mathrm{d}r\leq\int_a^b(\bar u_r'-\mathbf{u}_h')_+(r)\,\mathrm{d}r.$$

Rearranging this and using smallness of the r.h.s we get that

$$\bar u_r(a)-\bar u_r(b)\geq\mathbf{u}_h(a)-\mathbf{u}_h(b)-\text{small term}.$$

Since $\mathbf{u}_h(a)-\mathbf{u}_h(b)$ is bounded above away from 0, we see that either $\bar u_r(b)$ is below $-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}b-\delta_0 b$ or otherwise $\bar u_r(a)$ is much larger than $-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}a$. Since the latter can not be true for many points $a\in I_1$ (otherwise the last term on the first line in (3.15) becomes too large), we get the conclusion.

The rigorous argument goes as follows. First, we provide two estimates concerning $\mathbf{u}_h^0$, obtained from explicit computations. We recall that, for any $r\in[r_h,r_0]$, we have

$$\mathbf{u}_h^0(r)=-2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}r+g(r),$$

where $g(r):=-\frac16\left(\frac{r^3-r_h^3}{R^2}\right)+2\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}r_0\log\frac{r}{r_h}$. Note that, for any $s\in(r_h,r_0)$,

$$g'(s)=\frac{-s^3+4\alpha_s^{\frac12}h^{\frac{2-\beta}{2}}r_0R^2}{2sR^2}.$$

In particular, recalling that $r_h = (16\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 R^2)^{\frac{1}{3}}$, we obtain

$$g'(s) < \frac{-r_h^3 + 4\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 R^2}{2sR^2} < -6\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} < 0.$$

Since $g(r_h) = 0$, we conclude that

$$g(r) < 0 \quad \text{for any } r > r_h.$$

Moreover, recalling that $\alpha_s \le 2^{-8} 3^{-6} r_0^4 R^{-4}$ in the case $\beta = 2$, we see that, for any h sufficiently small (independently of β), we have

$$\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 R^2 \le 2^{-4} 3^{-3} r_0^3.$$

Hence, for any $s \in \left[\frac{1}{3} r_0, r_0\right)$,

$$g'(s) < \frac{-3^{-3} r_0^3 + 4\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_0 R^2}{2sR^2} \le -\frac{3^{-3} r_0^3 (1 - 2^{-2})}{2sR^2} \le -\frac{r_0^2}{2^3 3^2 R^2} =: -c_0.$$

Therefore, for any $r_1, r_2 \in \left[\frac{1}{3} r_0, r_0\right]$ with $r_1 < r_2$, we have

$$g(r_2) - g(r_1) = \int_{r_1}^{r_2} g'(s) \mathrm{d}s \le -c_0 (r_2 - r_1).$$

In particular, taking $r_1 = \frac{r_0}{3} \ge r_h$ and recalling that $g(r_1) \le g(r_h) = 0$, we are led to

$$g(r_2) \le -c_0 \left(r_2 - \frac{r_0}{3} \right).$$

We immediately deduce that, for a.e. $r_1, r_2 \in \left[\frac{1}{3} r_0, r_0\right]$ with $r_1 < r_2$,

$$\mathbf{u}_h^0(r_2) - \mathbf{u}_h^0(r_1) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} (r_2 - r_1) - c_0 (r_2 - r_1) \tag{3.32}$$

and

$$\mathbf{u}_h^0(r_2) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_2 - c_0 \left(r_2 - \frac{r_0}{3} \right).$$

From this last inequality, we deduce that, for a.e. $r \in \left[\frac{1}{2} r_0, r_0\right]$, we have

$$\frac{\mathbf{u}_h^0(r)}{r} \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \delta_1, \quad \text{where } \delta_1 = c_0 \left(1 - \frac{2}{3} \right) = \frac{c_0}{3}.$$

By (3.2), we deduce that

$$\frac{\mathbf{u}_h(r)}{r} \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \delta_1 + O\left(h^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \right) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \frac{1}{2}\delta_1, \tag{3.33}$$

provided h is sufficiently small (independently of β).

On the other hand, for a.e $r_1, r_2 \in \left[\frac{1}{2} r_0, r_0\right]$ with $r_1 < r_2$, we have that

$$(\bar{u}_r - \mathbf{u}_h)(r_2) - (\bar{u}_r - \mathbf{u}_h)(r_1) = \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')(r) \, \mathrm{d}r \le \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r) \, \mathrm{d}r.$$

In particular, for a.e. $r_1 \in I_1$ and a.e. $r_2 \in I_2$, we have that

$$\bar{u}_r(r_2) \le (\mathbf{u}_h(r_2) - \mathbf{u}_h(r_1)) + \bar{u}_r(r_1) + \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r) \, \mathrm{d}r.$$

From (3.2) and (3.32), we are led to

$$\mathbf{u}_h(r_2) - \mathbf{u}_h(r_1) \le \mathbf{u}_h^0(r_2) - \mathbf{u}_h^0(r_1) + O\left(h^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \right) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} (r_2 - r_1) - \frac{c_0}{2} (r_2 - r_1),$$

provided h is sufficiently small (independently of β). Therefore,

$$\bar{u}_r(r_2) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}(r_2 - r_1) - \frac{c_0}{2}(r_2 - r_1) + \bar{u}_r(r_1) + \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r)\,\mathrm{d}r. \tag{3.34}$$

From (3.15) and the upper bound construction, we deduce that

$$\int_{I_1} \left(\left(\frac{\mathbf{u}_h(r)}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \right) - \left(\frac{\bar{u}_r(r)}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \right) \right)^2 r\,\mathrm{d}r \le C h^{\frac{1}{4}},$$

provided h is sufficiently small (independently of β). Since $I_1 \subset \left[\frac{1}{2}r_0, r_0\right]$, from (3.33) we deduce that for a.e. $r \in I_1$,

$$\frac{\mathbf{u}_h(r)}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} = -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}.$$

Letting $\delta_2 := \frac{c_0}{2}\left(\frac{2}{3} - \frac{7}{12}\right) = \frac{c_0}{24}$ and

$$I_1^{\mathrm{bad}} := \left\{ r \in I_1 \mid \frac{\bar{u}_r(r)}{r} \ge -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} + \frac{1}{2}\delta_2 \right\},$$

we deduce that

$$\begin{aligned} |I_1^{\mathrm{bad}}| \left(\frac{1}{2}\delta_2\right)^2 \frac{1}{2} r_0 &\le \int_{I_1^{\mathrm{bad}}} \left(\frac{\bar{u}_r(r)}{r} + 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \right)^2 r\,\mathrm{d}r \\ &\le \int_{I_1} \left(\left(\frac{\mathbf{u}_h}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \right) - \left(\frac{\bar{u}_r}{r} \vee -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} \right) \right)^2 r\,\mathrm{d}r \le C h^{\frac{1}{4}}. \end{aligned}$$

In particular, provided h is sufficiently small (independently of β), we have

$$|I_1 \setminus I_1^{\mathrm{bad}}| \ge \frac{|I_1|}{2} = \frac{1}{24} r_0.$$

Then, by choosing $r_1 \in I_1 \setminus I_1^{\mathrm{bad}}$, we have

$$\bar{u}_r(r_1) \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_1 + \frac{1}{2}\delta_2 r_1 \le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_1 + \frac{1}{2}\delta_2 r_0.$$

Inserting this and $-\frac{c_0}{2}(r_2 - r_1) \le -\delta_2 r_0$ in (3.34), we find, for a.e. $r \in I_2$,

$$\begin{aligned} \bar{u}_r(r_2) &\le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}(r_2 - r_1) - \delta_2 r_0 - 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_1 + \frac{1}{2}\delta_2 r_0 + \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r)\,\mathrm{d}r \\ &= -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r_2 - \frac{1}{2}\delta_2 r_0 + \int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r)\,\mathrm{d}r, \end{aligned}$$

hence

$$\begin{aligned} \frac{\bar{u}_r(r_2)}{r_2} &\le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \frac{1}{2}\delta_2 + \frac{1}{r_2}\int_{r_1}^{r_2} (\bar{u}_r' - \mathbf{u}_h')_+(r)\,\mathrm{d}r \\ &\le -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} - \frac{1}{2}\delta_2 + \frac{3}{2r_0}\int_{\frac{1}{2}r_0}^{r_0} (\bar{u}_r' - \mathbf{u}_h')_+(r). \end{aligned} \tag{3.35}$$

To finish the argument, we combine (3.15) with the upper bound construction, which yields that

$$\frac{3}{2r_0}\int_{\frac{1}{2}r_0}^{r_0}(\bar{u}_r'-\mathbf{u}_h')_+(r)\le C\begin{cases}|\log h|^{\frac{c_2}{2}}h^{\frac{2+\beta}{8}} & \text{if } \frac{2}{3}\le\beta\le 2\\ |\log h|^{\frac{c_4}{2}}h^{\frac{\beta}{2}} & \text{if } 0<\beta\le\frac{2}{3}.\end{cases}$$

The right-hand side of this expression can be made smaller or equal than $\frac{\delta_2}{4}$, provided h is sufficiently small (independently of β only in the case $\frac{2}{3}<\beta\le 2$). Inserting this in (3.35), we obtain (3.31) with $\delta_0=\frac{1}{4}\delta_2$.

Step 2. Let $I_h=(a,b)$ be an interval of length $2h^{\frac{2+\beta}{8}}$ contained in I_2. Define $I_h^0=\left(a,a+\frac{1}{2}h^{\frac{2+\beta}{8}}\right)$ and $I_h^1=\left(b-\frac{1}{2}h^{\frac{2+\beta}{8}},b\right)$.

Let ρ_i be such that

$$W_{\rho_i}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i},w(\rho_i,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i}\right)=\min_{r\in I_h^i}W_r\left(\frac{\bar{u}_r(r)}{r},w(r,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(r)}{r}\right)$$

for $i=0,1$. Observe that $\lambda=\rho_1-\rho_0\in\left[\frac{|I_h|}{2},|I_h|\right)=[h^{\frac{2+\beta}{8}},2h^{\frac{2+\beta}{8}})$. The choice of ρ_0 and ρ_1 then implies

$$\begin{aligned}\int_{I_h}&\left[W_r\left(\frac{\bar{u}_r(r)}{r},w(r,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(r)}{r}\right)+\alpha_s^{\frac{1}{2}}h^{\frac{2-\beta}{2}}B(r)\right]\mathrm{d}r\\&\ge\frac{|I_h|}{4}\sum_{i=0}^{1}\left(W_{\rho_i}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i},w(\rho_i,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i}\right)\right)+\lambda\alpha_s^{\frac{1}{2}}h^{\frac{2-\beta}{2}}\fint_{\rho_0}^{\rho_1}B(r)\,\mathrm{d}r.\end{aligned}$$

Using (3.25), we deduce that

$$\int_{I_h}\left[W_r\left(\frac{\bar{u}_r(r)}{r},w(r,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(r)}{r}\right)+\alpha_s^{\frac{1}{2}}h^{\frac{2-\beta}{2}}B(r)\right]\mathrm{d}r\ge C_0|I_h|h^{\frac{6-\beta}{4}},$$

where C_0 denotes a constant that depends on α_s, r_0, and δ_0 only.

By covering the interval I_2 by intervals like I_h considered above, and using (3.4) (recall that $I_2\subset\left[\frac{1}{2}r_0,r_0\right]$), we deduce that

$$\int_{I_2}\left[W_r\left(\frac{\bar{u}_r(r)}{r},w(r,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(r)}{r}\right)+\sigma_hB(r)\right]\mathrm{d}r\ge C_0h^{\frac{6-\beta}{4}}.$$

Finally, combining the previous estimate with (3.15), we find the desired lower bound. □

Proof of the lower bound part of Theorem 2.2. We just need to modify Step 2 in the previous proof for $\beta\in[0,\frac{2}{3})$. Let $I_h=(a,b)$ be an interval of length $2h^{\frac{2+3\beta}{12}}$ contained in I_2. Define $I_h^0=\left(a,a+\frac{1}{2}h^{\frac{2+3\beta}{12}}\right)$ and $I_h^1=\left(b-\frac{1}{2}h^{\frac{2+3\beta}{12}},b\right)$.

Let ρ_i be such that

$$W_{\rho_i}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i},w(\rho_i,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i}\right)=\min_{r\in I_h^i}W_r\left(\frac{\bar{u}_r(r)}{r},w(r,\cdot)\right)-W_{\mathrm{rel}}\left(\frac{\bar{u}_r(r)}{r}\right)$$

for $i = 0, 1$. Observe that $\lambda = \rho_1 - \rho_0 \in \left[\frac{|I_h|}{2}, |I_h|\right) = [h^{\frac{2+3\beta}{12}}, 2h^{\frac{2+3\beta}{12}})$. Hence

$$\int_{I_h} \left[W_r\left(\frac{\bar{u}_r(r)}{r}, w(r,\cdot)\right) - W_{\rm rel}\left(\frac{\bar{u}_r(r)}{r}\right) + B^2(r)\right] \mathrm{d}r$$
$$\geq \frac{|I_h|}{4} \sum_{i=0}^{1} \left(W_{\rho_i}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i}, w(\rho_i,\cdot)\right) - W_{\rm rel}\left(\frac{\bar{u}_r(\rho_i)}{\rho_i}\right)\right) + \lambda \fint_{\rho_0}^{\rho_1} B^2(r)\,\mathrm{d}r.$$

Using (3.30), we deduce that

$$\int_{I_h} \left[W_r\left(\frac{\bar{u}_r(r)}{r}, w(r,\cdot)\right) - W_{\rm rel}\left(\frac{\bar{u}_r(r)}{r}\right) + B^2(r)\right] \mathrm{d}r \geq C_0 |I_h| h^{\frac{4}{3}},$$

where C_0 denotes a constant that depends on α_s, r_0, and δ_0 only.

By covering the interval I_2 by intervals like I_h considered above, we deduce that

$$\int_{I_2} \left[W_r\left(\frac{\bar{u}_r(r)}{r}, w(r,\cdot)\right) - W_{\rm rel}\left(\frac{\bar{u}_r(r)}{r}\right) + B^2(r)\right] \mathrm{d}r \geq C_0 h^{\frac{4}{3}}.$$

Combining with (3.16), we find the desired lower bound. □

4. Upper bound

We consider a configuration (u, w) of the form $\left(\left(\mathbf{u}_h^0(r) + u_r(r,\theta), u_\theta(r,\theta)\right), w(r,\theta)\right)$ with

$$\bar{u}_r(r) = \bar{u}_\theta(r) = \bar{w}_r(r) = 0 \quad \forall r \in (0, r_0).$$

In short, the functions u_r, u_θ, and w will account for the oscillations (wrinkles) expected to occur.

Observe that, for such a configuration, we have

(4.1)
$$E_h(u,w) - F_h^0(\mathbf{u}_h^0) = \int_0^{r_0} \left[2\sigma_h^0 B + B^2 + W_r\left(\frac{\mathbf{u}_h^0}{r}, w\right) - W_{\rm rel}\left(\frac{\mathbf{u}_h^0}{r}\right)\right] r\,\mathrm{d}r + R_h(u,\xi) + h^2 \frac{r_0^2}{2R^2},$$

where we recall that

$$B(r) = \fint_0^{2\pi} |\partial_r w(r,\theta)|^2\,\mathrm{d}\theta, \quad \sigma_h^0(r) = \mathbf{u}_h^0{}'(r) + \frac{r^2}{2R}, \quad \text{and} \quad \xi(r,\theta) = w(r,\theta) - \frac{r^2}{2R^2}.$$

Let us start our construction by defining $w(r,\theta)$. In what follows, we let $\ell, \delta > 0$ and $N := \dfrac{h^\delta}{\ell}$. Prompted by [6], we define

$$w(r,\theta) := A(r) r h^{\frac{\delta}{2}} \sum_{k>0} m\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r \ell}{h^{\frac{2+\beta}{4}}}\right] \frac{\sqrt{2}\cos(kN\theta)}{kN},$$

where $A(r)$ is a non-negative function supported in $[r_h, r_0]$ that will be defined below and

$$m(t) := \begin{cases} \exp\left(-\dfrac{1}{1-4|t|^2}\right) & \text{if } |t| \leq \frac{1}{2}, \\ 0 & \text{if } |t| > \frac{1}{2} \end{cases}$$

is a classical smooth bump function. In order to simplify the notation, in what follows we write $\sum_k := \sum_{k>0}$.

By the support condition on A, we immediately see that, for any $r \in [0, r_h]$,

$$W_r\left(\frac{\mathbf{u}_h^0}{r}, w(r,\cdot)\right) - W_{\mathrm{rel}}\left(\frac{\mathbf{u}_h^0}{r}\right) = 0.$$

On the other hand, for any $r \in (r_h, r_0]$ we have that $\mathbf{u}_h^0(r) < -2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}} r$, and therefore (3.20) yields

$$W_r\left(\frac{\mathbf{u}_h^0}{r}, w(r,\cdot)\right) - W_{\mathrm{rel}}\left(\frac{\mathbf{u}_h^0}{r}\right) = |\gamma(r) - \gamma_0(r)|^2 + A^2(r) h^\delta \sum_k m^2\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right]\left(\frac{hNk}{r} - \frac{\alpha^{\frac{1}{2}} r}{h^{\frac{\beta}{2}} Nk}\right)^2 \tag{4.2}$$

with

$$\gamma(r) := A^2(r) h^\delta \sum_k m^2\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \quad \text{and} \quad \gamma_0(r) := -\left(\frac{\mathbf{u}_h^0}{r} + 2\alpha_s^{\frac{1}{2}} h^{\frac{2-\beta}{2}}\right) > 0.$$

Basically, $\gamma(r)$ is the amount of excess length produced by wrinkles, while $\gamma_0(r)$ is the ideal excess length amount dictated by the solution $\mathbf{u}_h^0$ of the relaxed problem. We will now define A in such a way that this difference is small. We let η be a smooth cut-off function such that $\eta(t) = 1$ if $t > 2$ and $\eta(t) = 0$ if $t < 1$, and define

$$A(r) := \begin{cases} 0 & r \in (0, r_h) \\ \eta\left(\dfrac{r - r_h}{h^\alpha}\right)\left(\dfrac{\gamma_0(r)}{\int_{\mathbb{R}} m^2}\right)^{\frac{1}{2}} & r \in (r_h, r_0), \end{cases}$$

where $\alpha > 0$ is such that $h^\alpha \ll r_h$, and will be determined afterwards. A direct (but tedious) computation shows that

$$|A'(r)| \le C\left(h^{-\alpha} r_h\right)^{\frac{1}{2}} \quad \text{and} \quad |A''(r)| \le C\left(h^{-3\alpha} r_h\right)^{\frac{1}{2}}. \tag{4.3}$$

Letting

$$\tilde{\gamma}(r) := A^2(r) h^\delta \int_{\mathbb{R}} m^2\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \mathrm{d}k = A^2(r) \int_{\mathbb{R}} m^2,$$

observe that by definition of A we have $\tilde{\gamma}(r) = \gamma_0(r)$ for any $r \in (r_h + 2h^\alpha, r_0)$, and

$$|\tilde{\gamma}(r) - \gamma_0(r)| \le C h^\alpha r_h$$

for any $r \in [r_h, r_h + 2h^\alpha]$, which implies that

$$\int_{r_h}^{r_0} |\tilde{\gamma}(r) - \gamma_0(r)|^2 r \,\mathrm{d}r \le C(h^\alpha r_h)^3. \tag{4.4}$$

Let us now estimate the right-hand side (4.2), starting with the first term. For that, we make the following observation: for any smooth compactly supported function $f : \mathbb{R} \to \mathbb{R}$ there exists a constant $C_f > 0$, depending only on the support of f such that, for any $0 \neq n \in \mathbb{N}$, $t \in (0,1)$, and any shift $\zeta \in \mathbb{R}$, one has

$$\left| t \sum_{k \in \mathbb{Z}} f(tk + \zeta) - \int_{\mathbb{R}} f \right| \le C_f t^n \|f^{(n)}\|_\infty. \tag{4.5}$$

Indeed, first observe that there exists $\zeta' \in [0,1)$ such that $\int_{\mathbb{R}} f = t\sum_{k\in\mathbb{Z}} f(tk+\zeta')$, which by the mean-value property implies

$$t\left|\sum_k f(tk+\zeta) - t^{-1}\int_{\mathbb{R}} f\right| = t\left|\sum_k f(tk+\zeta) - \sum_k f(tk+\zeta')\right| \le t\left|t\sum_k f'(tk+\zeta_k'')\right|.$$

Since $\int_{\mathbb{R}} f^{(m)} = 0$ for $m \ge 1$, in particular $\left|t\sum_k f^{(m)}(tk+\zeta_k'')\right| = \left|t\sum_k f^{(m)}(tk+\zeta_k'') - \int_{\mathbb{R}} f^{(m)}\right|$, iterating the previous estimate with f replaced inductively with $f^{(m)}$, we get that

$$\left|t\sum_k f(tk+\zeta) - \int_{\mathbb{R}} f\right| \le t^m\left|t\sum_k f^{(m)}(tk+\zeta'')\right|$$

for some $\zeta'' \in [0,1)$ and any $m \in \mathbb{N}$. To conclude, we observe that $\#\left\{k \mid f^{(m)}(tk+\zeta'') \neq 0\right\} \le C_f t^{-1}$, where C_f is a constant that depends only on the support of f, and the claim follows from

$$\left|t\sum_k f^{(m)}(tk+\zeta'')\right| \le C_f\|f^{(m)}\|_\infty.$$

On the other hand, a direct computation shows that, for any $n \in \mathbb{N}$,

$$\|(m^2)^{(n)}\|_\infty \le (C_m n)^{(C_m' n)},$$

where hereafter C_m and C_m' denote constants only depending on m that may change from line to line. By combining this with (4.5) for $f = m^2$, we are led to

$$|\gamma(r) - \tilde{\gamma}(r)| \le (C_m n)^{(C_m' n)}(h^\delta)^n$$

for any $n \in \mathbb{N}$. Optimizing over n on the right-hand side, that is choosing

$$n = (C_m e^{\frac{1}{C_m'}})^{-1} h^{-\frac{\delta}{C_m'}}$$

yields

$$\int_{r_h}^{r_0} |\gamma(r) - \tilde{\gamma}(r)|^2 r\,\mathrm{d}r \le C\exp\left(-\frac{h^{-\frac{\delta}{C_m'}}}{C_m}\right)^2.$$

This combined with (4.4), gives

$$\int_{r_h}^{r_0} |\gamma(r) - \gamma_0(r)|^2 r\,\mathrm{d}r \le C\left((h^\alpha r_h)^3 + \exp\left(-\frac{h^{-\frac{\delta}{C_m'}}}{C_m}\right)^2\right). \tag{4.6}$$

Let us now estimate the second term on the right-hand side of (4.2). We define

$$g(k) := \frac{hNk}{r} - \frac{\alpha^{\frac{1}{2}} r}{h^{\frac{\beta}{2}} Nk}.$$

Observe that $g(k_0) = 0$, where $k_0 := \frac{\alpha^{\frac{1}{4}} r}{h^{\frac{2+\beta}{4}} N}$ is the frequency k for which

$$m\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] = m(0) = 1.$$

Also, $g'(k_0)=\frac{2hN}{r}$. By the support condition on m, the sum $\sum_k m\left[h^\delta k-\frac{\alpha^{\frac14}r\ell}{h^{\frac{2+\beta}{4}}}\right]^2$ is performed over $O(h^{-\delta})$ frequencies k such that $|k-k_0|\le\frac12 h^{-\delta}$, and therefore

$$h^\delta\sum_k m\left[h^\delta k-\frac{\alpha^{\frac14}r\ell}{h^{\frac{2+\beta}{4}}}\right]^2 g(k)^2\le Ch^\delta h^{-\delta}\left(g'(k_0)h^{-\delta}\right)^2\le\frac{C}{r^2}\frac{h^2}{\ell^2}.$$

Besides, using (2.3) one can check that $\frac{A^2(r)}{r^2}$ is bounded. Thus

$$\int_0^{r_0}A^2(r)h^\delta\sum_k m^2\left[h^\delta k-\frac{\alpha^{\frac14}r\ell}{h^{\frac{2+\beta}{4}}}\right]\left(\frac{hNk}{r}-\frac{\alpha^{\frac12}r}{h^{\frac{\beta}{2}}Nk}\right)^2 r\,\mathrm{d}r\le C\frac{h^2}{\ell^2},$$

which combined with (4.6) gives

$$\int_0^{r_0}\left[W_r\left(\frac{\mathbf{u}_h^0}{r},w\right)-W_{\mathrm{rel}}\left(\frac{\mathbf{u}_h^0}{r}\right)\right]r\,\mathrm{d}r\le C\left((h^\alpha r_h)^3+\exp\left(-h^{-\frac{\delta}{C_m'}}\right)^2+\frac{h^2}{\ell^2}\right). \tag{4.7}$$

We will now estimate $\int_0^{r_0}(\sigma_h^0B+B^2)r\,\mathrm{d}r$. By definition

$$B(r)=\fint_0^{2\pi}|\partial_r w(r,\theta)|^2\,\mathrm{d}\theta=h^\delta\sum_k\left(\partial_r\left(A(r)rm\left[h^\delta k-\frac{\alpha^{\frac14}r\ell}{h^{\frac{2+\beta}{4}}}\right]\right)\right)^2\frac{1}{(kN)^2}.$$

Using once again the fact that the sum is performed over $O(h^{-\delta})$ frequencies k such that $|k-k_0|\le\frac12 h^{-\delta}$, which will be used repeatedly hereafter, we deduce that for these frequencies we have

$$kN\ge C\frac{r}{h^{\frac{2+\beta}{4}}}. \tag{4.8}$$

Thus, using (4.3) and noting that $\frac{A^2(r)}{r^2}\le C$, we find

$$B(r)\le Ch^{\frac{2+\beta}{2}}\left(|A'(r)|^2+\frac{\ell^2}{h^{\frac{2+\beta}{2}}}\right)\le C(h^{\frac{2+\beta}{2}}h^{-\alpha}r_h+\ell^2).$$

Recalling that $0\le\sigma_h^0r\le Ch^{\frac{2-\beta}{2}}$ for $r\ge r_h$, we find that

$$\int_0^{r_0}\sigma_h^0B(r)r\,\mathrm{d}r\le C(h^{2-\alpha}r_h+h^{\frac{2-\beta}{2}}\ell^2). \tag{4.9}$$

Besides,

$$\int_0^{r_0}B^2(r)r\,\mathrm{d}r\le C(h^{2+\beta-2\alpha}r_h^2+\ell^4). \tag{4.10}$$

We now proceed to define $u_r(r,\theta)$ and $u_\theta(r,\theta)$ and estimate $R_h(u,\xi)$, which we write as the sum of the following five terms:

$$R_h^1 = \int_0^{r_0} \fint_0^{2\pi} \left| \partial_r u_r + \frac{(\partial_r \xi)^2}{2} - \frac{\overline{(\partial_r \xi)^2}}{2} \right|^2 \,\mathrm{d}\theta r \,\mathrm{d}r,$$

$$R_h^2 = \int_0^{r_0} \fint_0^{2\pi} \left| \frac{\partial_\theta u_\theta}{r} + \frac{u_r}{r} + \frac{(\partial_\theta w)^2}{2r^2} - \frac{\overline{(\partial_\theta w)^2}}{2r^2} \right|^2 \,\mathrm{d}\theta r \,\mathrm{d}r,$$

$$R_h^3 = \int_0^{r_0} \fint_0^{2\pi} \frac{1}{2} \left| \frac{\partial_\theta u_r}{r} + r\partial_r \left(\frac{u_\theta}{r}\right) + \frac{1}{r}\partial_r \xi \partial_\theta \xi \right|^2 \,\mathrm{d}\theta r \,\mathrm{d}r,$$

$$R_h^4 = \int_0^{r_0} \fint_0^{2\pi} h^2 \left|\partial_{rr} w\right|^2 \,\mathrm{d}\theta r \,\mathrm{d}r,$$

$$R_h^5 = \int_0^{r_0} \fint_0^{2\pi} \frac{2h^2}{r^2} \left|\partial_{\theta r} \xi\right|^2 \,\mathrm{d}\theta r \,\mathrm{d}r.$$

Let us begin by estimating the terms which do not depend on u, that is R_h^4 and R_h^5.

Estimating R_h^4. We have that

$$\partial_{rr} w = h^{\frac{\delta}{2}} \sum_k \partial_{rr} \left(A(r) r m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r \ell}{h^{\frac{2+\beta}{4}}} \right] \right) \frac{\sqrt{2}\cos(kN\theta)}{kN}.$$

Using (4.8) and $\frac{|A'(r)|}{r} \le C|A''(r)|$, we deduce that

$$|\partial_{rr} w| \le C h^{\frac{\delta}{2}} h^{-\delta} h^{\frac{2+\beta}{4}} \left(|A''(r)| + \frac{\ell^2}{h^{\frac{2+\beta}{2}}} \right),$$

which combined with (4.3) yields

$$R_h^4 \le C h^{-\delta} \left(h^{\frac{2+\beta}{2}} h^{2-3\alpha} r_h + \ell^4 h^{\frac{2-\beta}{2}} \right). \tag{4.11}$$

Estimating R_h^5. We have that

$$\partial_{\theta r} \xi = -h^{\frac{\delta}{2}} \sum_k \partial_r \left(A(r) r m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r \ell}{h^{\frac{2+\beta}{4}}} \right] \right) \sqrt{2}\sin(kN\theta)$$

Therefore

$$\frac{|\partial_{\theta r} \xi|}{r} \le C h^{\frac{\delta}{2}} h^{-\delta} \left(|A'(r)| + \frac{\ell}{h^{\frac{2+\beta}{4}}} \right),$$

which combined with (4.3) implies that

$$R_h^5 \le C h^{-\delta} \left(h^{2-\alpha} r_h + \ell^2 h^{\frac{2-\beta}{2}} \right). \tag{4.12}$$

Defining u_r and u_θ. For this, let us first analyze R_h^2. We begin by observing that

$$(\partial_\theta w)^2 = A^2(r) h^\delta \sum_{k,j} m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r \ell}{h^{\frac{2+\beta}{4}}} \right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r \ell}{h^{\frac{2+\beta}{4}}} \right] \sin(kN\theta)\sin(jN\theta)$$

and

$$\overline{(\partial_\theta w)^2} = \frac{1}{2}A^2(r)h^\delta \sum_k m^2 \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right].$$

Writing

$$\sin(kN\theta)\sin(jN\theta) = -\frac{1}{2}\cos((k+j)N\theta)) + \frac{1}{2}\cos((k-j)N\theta)),$$

we obtain

$$\begin{aligned}(\partial_\theta w)^2 - \overline{(\partial_\theta w)^2} = &- \frac{1}{2}A^2(r)h^\delta \sum_{k,j} m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \cos((k+j)N\theta)) \\ &+ \frac{1}{2}A^2(r)h^\delta \sum_{k\neq j} m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \cos((k-j)N\theta)).\end{aligned}$$

Since these terms are of order $O(h^{-\delta})$, we will define u_θ in such a way that they cancel. More precisely, we let

$$u_\theta(r,\theta) = \frac{1}{2}(u_{\theta,+} - u_{\theta,-}),$$

where

$$\begin{aligned}u_{\theta,+}(r,\theta) &:= A^2(r)rh^\delta \sum_{k,j} m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \frac{\sin((k+j)N\theta)}{(k+j)N}, \\ u_{\theta,-}(r,\theta) &:= A^2(r)rh^\delta \sum_{k\neq j} m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right] \frac{\sin((k-j)N\theta)}{(k-j)N},\end{aligned}$$

Note that

$$\frac{\partial_\theta u_\theta}{r} + \frac{(\partial_\theta w)^2}{2r^2} - \frac{\overline{(\partial_\theta w)^2}}{2r^2} = 0,$$

and therefore

$$R_h^2 = \int_0^{r_0} \fint_0^{2\pi} \left|\frac{u_r}{r}\right|^2 \,\mathrm{d}\theta r\,\mathrm{d}r. \tag{4.13}$$

Let us now analyze R_h^1. Observe that

$$(\partial_r \xi)^2 = \left(\partial_r w - \frac{r}{R}\right)^2 = (\partial_r w)^2 - 2\frac{r}{R}\partial_r w + \frac{r^2}{R^2},$$

and since $\overline{\partial_r w} = 0$, we have

$$\frac{1}{2}\left((\partial_r\xi)^2 - \overline{(\partial_r\xi)^2}\right) = \frac{1}{2}\left((\partial_r w)^2 - B\right) - \frac{r}{R}\partial_r w. \tag{4.14}$$

Let us estimate the first term on the right-hand side. Letting

$$g_k(r) := A(r)rm\left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}}\right],$$

we have

$$(\partial_r w)^2 = 2h^\delta \sum_{k,j} \partial_r g_k(r)\partial_r g_j(r)\frac{\cos(kN\theta)\cos(jN\theta)}{kjN^2}.$$

Arguing as when estimating B, we find

$$|(\partial_r w)^2| \le Ch^\delta h^{-2\delta} h^{\frac{2+\beta}{2}} \left(|A'|^2 + \frac{\ell^2}{h^{\frac{2+\beta}{2}}} \right) \le Ch^{-\delta}(h^{\frac{2+\beta}{2}} h^{-\alpha} r_h + \ell^2).$$

Therefore

$$\int_0^{r_0} \fint_0^{2\pi} \left| \frac{1}{2}\left((\partial_r w)^2 - B\right) \right|^2 \, \mathrm{d}\theta r \, \mathrm{d}r \le Ch^{-2\delta}(h^{2+\beta-2\alpha} r_h^2 + \ell^4). \tag{4.15}$$

Conversely, the second term on the right-hand side of (4.14) is too large. For this reason, we let

$$u_r(r,\theta) := u_{r,1}(r,\theta) + u_{r,2}(r,\theta) \quad \text{with } \bar{u}_{r,1}(r) = \bar{u}_{r,2}(r) = 0 \quad \forall r \in [0, r_0],$$

where

$$u_{r,1}(r,\theta) := \frac{r}{R} w(r,\theta)$$

and $u_{r,2}$ will be defined later. Note that

$$\partial_r u_{r,1} - \frac{r}{R}\partial_r w = \frac{w}{R}.$$

It is worth mentioning that our choice of $u_{r,1}$ cancels the $O(1)$-term $\frac{r}{R}\partial_\theta w$ which appears in R_h^3 (see (4.17) below).

Note that

$$\int_0^{r_0} \fint_0^{2\pi} \left| \partial_r u_{r,1} - \frac{r}{R}\partial_r w \right|^2 \, \mathrm{d}\theta r \, \mathrm{d}r = \int_0^{r_0} \fint_0^{2\pi} \left| \frac{w}{R} \right|^2 \, \mathrm{d}\theta r \, \mathrm{d}r \le Ch^{-\delta} h^{\frac{2+\beta}{2}},$$

which combined with (4.15) yields

$$R_h^1 \le \int_0^{r_0} \fint_0^{2\pi} |\partial_r u_{r,2}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r + Ch^{-2\delta}(h^{2+\beta-2\alpha} r_h^2 + \ell^4 + h^{\frac{2+\beta}{2}}). \tag{4.16}$$

We now turn to R_h^3. First, we observe that

$$\partial_\theta u_{r,1} + \partial_r \xi \partial_\theta \xi = \frac{r}{R}\partial_\theta w + \partial_r w \partial_\theta w - \frac{r}{R}\partial_\theta w = \partial_r w \partial_\theta w. \tag{4.17}$$

This term and $\partial_r \left(\frac{u_\theta}{r}\right)$ are too large. For this reason we will define $u_{r,2}$ in such a way that

$$\frac{\partial_\theta u_{r,2}}{r} + r\partial_r \left(\frac{u_\theta}{r}\right) + \frac{1}{r}\partial_r w \partial_\theta w = 0,$$

which immediately implies that $R_h^3 = 0$.

First, we let $U_r(r,\theta) = \frac{1}{2}(U_{r,+} - U_{r,-})$, where

$$U_{r,+}(r,\theta) := r^2 h^\delta \sum_{k,j} \partial_r \left(A^2(r) m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}} \right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}} \right] \right) \frac{\cos((k+j)N\theta)}{(k+j)^2 N^2},$$

$$U_{r,-}(r,\theta) := r^2 h^\delta \sum_{k\neq j} \partial_r \left(A^2(r) m \left[h^\delta k - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}} \right] m \left[h^\delta j - \frac{\alpha^{\frac{1}{4}} r\ell}{h^{\frac{2+\beta}{4}}} \right] \right) \frac{\cos((k-j)N\theta)}{(k-j)^2 N^2},$$

and note that

$$\frac{\partial_\theta U_r}{r} + r\partial_r \left(\frac{u_\theta}{r}\right) = 0.$$

Second, writing

$$2\cos(kN\theta)\sin(jN\theta) = \sin((k+j)N\theta) - \sin((k-j)N\theta),$$

and defining $M_k(r) := A(r) r m \left[h^\delta k - \frac{\alpha^{\frac14} r\ell}{h^{\frac{2+\beta}{4}}}\right]$, we have

$$\begin{aligned}\partial_r w \partial_\theta w(r,\theta) = &- h^\delta \sum_{k,j} (\partial_r M_k(r)) M_j(r) \frac{\sin((k+j)N\theta))}{kN} \\ &+ h^\delta \sum_{k\neq j} (\partial_r M_k(r)) M_j(r) \frac{\sin((k-j)N\theta))}{kN}.\end{aligned}$$

Thus, letting $V_r(r,\theta) = V_{r,+} - V_{r,-}$, where

$$\begin{aligned}V_{r,+}(r,\theta) &:= -h^\delta \sum_{k,j} (\partial_r M_k(r)) M_j(r) \frac{\cos((k+j)N\theta)}{k(k+j)N^2}, \\ V_{r,-}(r,\theta) &:= -h^\delta \sum_{k\neq j} (\partial_r M_k(r)) M_j(r) \frac{\cos((k-j)N\theta)}{k(k-j)N^2},\end{aligned}$$

we have

$$\partial_\theta V_r + \partial_r w \partial_\theta w = 0.$$

We finally set $u_{r,2} := U_r + V_r$.

Estimating R_h^1, R_h^2, and R_h^3. Recalling (4.13), is not hard to see that

$$R_h^2 \le C \int_0^{r_0} \fint_0^{2\pi} |\partial_r u_{r,2}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r,$$

which combined with (4.16) and $R_h^3 = 0$, implies

$$R_h^1 + R_h^2 + R_h^3 \le C \int_0^{r_0} \fint_0^{2\pi} |\partial_r u_{r,2}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r + C h^{-2\delta}(h^{2+\beta-2\alpha} r_h^2 + \ell^4 + h^{\frac{2+\beta}{2}}). \tag{4.18}$$

In addition, it is easy to see that

$$\int_0^{r_0} \fint_0^{2\pi} |\partial_r u_{r,2}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r \le \int_0^{r_0} \fint_0^{2\pi} |\partial_r (U_{r,-} + V_{r,-})|^2 \, \mathrm{d}\theta r \, \mathrm{d}r$$

Let us start by estimating $\int_0^{r_0} \fint_0^{2\pi} |\partial_r U_{r,-}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r$. We define

$$H(r) := h^\delta \sum_{k\neq j} m \left[h^\delta k - \frac{\alpha^{\frac14} r\ell}{h^{\frac{2+\beta}{4}}}\right] m \left[h^\delta j - \frac{\alpha^{\frac14} r\ell}{h^{\frac{2+\beta}{4}}}\right] \frac{\cos((k-j)N\theta)}{(k-j)^2 N^2}.$$

By definition, we have

$$\partial_r U_{r,-}(r,\theta) = \partial_r(r^2 \partial_r(A^2(r)))H(r) + 2[r^2\partial_r(A^2(r)) + rA^2(r)]\partial_r H(r) + r^2 A^2(r) \partial_{rr} H(r).$$

First, a straightforward computation shows that

$$|r^2 \partial_r(A^2(r)) + rA^2(r)| + |r^2 A^2(r)| \le C$$

and

$$|\partial_r(r^2\partial_r(A^2(r)))| \le C(1 + h^{-\alpha} r_h^3) \le C(1 + h^{\frac{2-\beta}{2}} h^{-\alpha}).$$

Thus

$$|\partial_r U_{r,-}(r,\theta)| \le C(1 + h^{\frac{2-\beta}{2}} h^{-\alpha})|H(r)| + C\left(|\partial_r H(r)| + |\partial_{rr} H(r)|\right). \tag{4.19}$$

Next, we note that

$$|H(r)| \leq \frac{Ch^\delta}{N^2}. \tag{4.20}$$

In addition, we observe that $H(r)$ is $\frac{h^{\frac{2+\beta}{4}}}{\alpha^{\frac{1}{4}}\ell}h^\delta$-periodic. Then, a direct computation shows that, for any $n \in \mathbb{N}$,

$$\|m^{(n)}\|_\infty \leq (C_m n)^{(C'_m n)},$$

which allows us to deduce that

$$\|H^{(n)}\|_\infty \leq (C_m n)^{(C'_m n)} \frac{h^\delta}{N^2} \left(\frac{\alpha^{\frac{1}{4}}\ell}{h^{\frac{2+\beta}{4}}}\right)^n.$$

By observing that, for any $n \geq 2$, $H^{(n-1)}$ has mean zero, due to the periodicity of $H^{(n-2)}$, we deduce that there exists $t_{n-1} \in \left(0, \frac{h^{\frac{2+\beta}{4}}}{\alpha^{\frac{1}{4}}\ell}h^\delta\right)$ such that $H^{(n-1)}(t_{n-1}) = 0$. Therefore, by Taylor expansion, for any $s \in \left(0, \frac{h^{\frac{2+\beta}{4}}}{\alpha^{\frac{1}{4}}\ell}h^\delta\right)$, we have

$$|H^{(n-1)}(s)| \leq \|H^{(n)}\|_\infty \frac{\alpha^{\frac{1}{4}}\ell}{h^{\frac{2+\beta}{4}}}h^\delta \leq (C_m n)^{(C'_m n)} \frac{h^\delta}{N^2} \left(\frac{\alpha^{\frac{1}{4}}\ell}{h^{\frac{2+\beta}{4}}}\right)^n \frac{h^{\frac{2+\beta}{4}}}{\alpha^{\frac{1}{4}}\ell}h^\delta.$$

Iterating this procedure, we obtain

$$\begin{aligned}|H'| + |H''| &\leq (C_m n)^{(C'_m n)} \frac{1}{N^2} \left(\frac{\alpha^{\frac{1}{4}}\ell}{h^{\frac{2+\beta}{4}}}\right)^2 h^{\delta(n-1)} \\ &\leq (C_m n)^{(C'_m n)} \left(\frac{\ell^2}{h^{\frac{2+\beta}{4}}}\right)^2 h^{\delta(n-3)}.\end{aligned}$$

Optimizing over n on the right-hand side, that is, choosing

$$n = (C_m e^{\frac{1}{C'_m}})^{-1} h^{-\frac{\delta}{C'_m}}$$

yields

$$|H'| + |H''| \leq C \left(\frac{\ell^2}{h^{\frac{2+\beta}{4}}}\right)^2 h^{-3\delta} \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right).$$

Plugging in this and (4.20) into (4.19), yields

$$\int_0^{r_0} \fint_0^{2\pi} |\partial_r U_{r,-}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r \leq C(1 + h^{2-\beta-2\alpha})\ell^4 h^{-2\delta} + C \left(\frac{\ell^2}{h^{\frac{2+\beta}{4}}}\right)^4 h^{-6\delta} \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2.$$

We now estimate $\int_0^{r_0} \fint_0^{2\pi} |\partial_r V_{r,-}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r$. Arguing similarly as above, once deduces that

$$|\partial_r V_{r,-}| \leq C \left((Ar)^2 |\partial_{rr} H| + |\partial_r((Ar)^2)||\partial_r H| + |\partial_{rr}((Ar)^2)||H|\right).$$

Using that

$$|(Ar)^2| + |\partial_r((Ar)^2)| \leq C, \quad |\partial_{rr}((Ar)^2)| \leq C(1 + h^{\frac{2-\beta}{2}} h^{-\alpha}),$$

and the estimates on H and its derivatives, we deduce that

$$\int_0^{r_0} \fint_0^{2\pi} |\partial_r V_{r,-}|^2 \, \mathrm{d}\theta r \, \mathrm{d}r \leq C(1+h^{2-\beta-2\alpha})\ell^4 h^{-2\delta} + C\left(\frac{\ell^2}{h^{\frac{2+\beta}{4}}}\right)^4 h^{-6\delta} \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2.$$

From (4.18), we deduce that

$$R_h^1 + R_h^2 + R_h^3 \leq C(1+h^{2-\beta-2\alpha})\ell^4 h^{-2\delta} + C\left(\frac{\ell^2}{h^{\frac{2+\beta}{4}}}\right)^4 h^{-6\delta} \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2 + Ch^{-2\delta}(h^{2+\beta-2\alpha} r_h^2 + \ell^4 + h^{\frac{2+\beta}{2}}). \tag{4.21}$$

Choice of the parameters α, ℓ, and δ : The case $\frac{2}{3} \leq \beta \leq 2$. In order to choose ℓ, following the same strategy as in the proof of the lower bound, we match the terms that depend on ℓ on the right-hand sides of (4.7) and (4.9), that is

$$\frac{h^2}{\ell^2} = h^{\frac{2-\beta}{2}} \ell^2,$$

which gives $\ell = h^{\frac{2+\beta}{8}}$. In particular, this implies that

$$\frac{h^2}{\ell^2} = h^{\frac{2-\beta}{2}} \ell^2 = h^{\frac{6-\beta}{4}}.$$

Let us now choose α. For this, we consider the largest term depending on α in the estimate for R_h – after a quick inspection, one realizes that this corresponds to $h^{2-\beta-2\alpha}\ell^4 h^{-2\delta}$ – and define α in such a way that (up to a constant)

$$h^{2-\beta-2\alpha}\ell^4 = h^{\frac{6-\beta}{4}},$$

that is $\alpha = \frac{6-\beta}{8}$. In particular, notice that this choice ensures that $h^\alpha \ll r_h$ and

$$(h^\alpha r_h)^3 + h^{2-\alpha} r_h + h^{\frac{2+\beta}{2}} h^{2-3\alpha} r_h + h^{2+\beta-2\alpha} r_h^2 \leq C h^{\frac{6-\beta}{4}}.$$

Therefore, we deduce from (4.7), (4.9), and (4.10) that

$$\int_0^{r_0} \left[2\sigma_h^0 B + B^2 + W_r\left(\frac{\mathbf{u}_h^0}{r}, w\right) - W_{\mathrm{rel}}\left(\frac{\mathbf{u}_h^0}{r}\right)\right] r \, \mathrm{d}r \leq C\left(h^{\frac{6-\beta}{4}} + \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2\right). \tag{4.22}$$

On the other hand, from (4.11), (4.12), and (4.21), we deduce that

$$R_h \leq C\left(h^{-2\delta} h^{\frac{6-\beta}{4}} + h^{-6\delta} \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2\right). \tag{4.23}$$

Finally, we choose δ in such a way that

$$h^{\frac{6-\beta}{4}} = \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2.$$

This implies

$$h^{-\delta} = \left(C_m \log \frac{1}{h^{\frac{6-\beta}{8}}}\right)^{C'_m},$$

that is

$$\delta = \frac{\log\left(C_m \log \frac{1}{h^{\frac{6-\beta}{8}}}\right)^{C'_m}}{\log \frac{1}{h}}.$$

Hence, inserting this into (4.22) and (4.23), and combining with (4.1), we obtain

$$E_h(u,w) - F_h^0(\mathbf{u}_h^0) \leq \left(C_m \log \frac{1}{h^{\frac{6-\beta}{8}}}\right)^{6C'_m} h^{\frac{6-\beta}{4}}.$$

This concludes the proof of the upper bound in the case $\frac{2}{3} \leq \beta \leq 2$.

Choice of the parameters α, ℓ, and δ : The case $0 \leq \beta < \frac{2}{3}$. Since $\sigma_h^0 \geq 0$, following once again the proof of the lower bound, in order to choose ℓ we match the terms that depend on ℓ on the right-hand sides of (4.7) and (4.10), that is

$$\frac{h^2}{\ell^2} = \ell^4,$$

which gives $\ell = h^{\frac{1}{3}}$. In particular, this implies that

$$\frac{h^2}{\ell^2} = \ell^4 = h^{\frac{4}{3}}.$$

To define α, we consider once again the largest term depending on α in the estimate for R_h – after a quick inspection, one realizes that this corresponds to $h^{\frac{2+\beta}{2}} h^{2-3\alpha} r_h$ – and define α in such a way that (up to a constant)

$$h^{\frac{2+\beta}{2}} h^{2-3\alpha} r_h = h^{\frac{2+\beta}{2}},$$

that is $\alpha = \frac{10-\beta}{18}$. In particular, notice that this choice ensures that $h^\alpha \ll r_h$ and

$$(h^\alpha r_h)^3 + h^{2-\alpha} r_h + h^{2-\beta-2\alpha}\ell^4 + h^{2+\beta-2\alpha} r_h^2 \leq C h^{\frac{2+\beta}{2}}.$$

We then deduce from (4.7), (4.9), and (4.10) that

$$\int_0^{r_0} \left[2\sigma_h^0 B + B^2 + W_r\left(\frac{\mathbf{u}_h^0}{r}, w\right) - W_{\mathrm{rel}}\left(\frac{\mathbf{u}_h^0}{r}\right)\right] r\,\mathrm{d}r \leq C\left(h^{\frac{2+\beta}{2}} + \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2\right).$$

On the other hand, from (4.11), (4.12), and (4.21), we deduce that

$$R_h \leq C\left(h^{-2\delta} h^{\frac{2+\beta}{2}} + h^{-6\delta}\exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2\right).$$

Let us note that the term $h^{\frac{2+\beta}{2}}$, which for $\beta < \frac{2}{3}$ dominates $h^{\frac{4}{3}}$, essentially comes from the term $|w|^2$ in R_h^1. We were not able to get rid of it, and we believe that in the case $0 \leq \beta < \frac{2}{3}$ it is indeed the leading order term in the expansion of $E_h(u,w) - F_h^0(\mathbf{u}_h^0)$. Since in our upper bound construction the only place where it appears is in R_h, in order to provide a matching lower bound one would need to extract, for a minimizing configuration, a term of the same order from R_h, which we do not know how to do.

It is worth noticing that $h^{\frac{2+\beta}{2}} = h^{\frac{4}{3}}$ for $\beta = \frac{2}{3}$ and that $h^{\frac{2+\beta}{2}} \ll h^{\frac{6-\beta}{4}}$ if $\frac{2}{3} < \beta \leq 2$, reason for which this term is of lower order in the previously considered case.

Finally, we choose δ in such a way that

$$h^{\frac{2+\beta}{2}} = \exp\left(-\frac{h^{-\frac{\delta}{C'_m}}}{C_m}\right)^2.$$

This implies

$$h^{-\delta} = \left(C_m \log \frac{1}{h^{\frac{2+\beta}{4}}}\right)^{C'_m},$$

that is

$$\delta = \frac{\log\left(C_m \log \frac{1}{h^{\frac{2+\beta}{4}}}\right)^{C'_m}}{\log \frac{1}{h}}.$$

Hence, we deduce that

$$E_h(u,w) - F_h^0(\mathbf{u}_h^0) \leq \left(C_m \log \frac{1}{h^{\frac{2+\beta}{4}}}\right)^{C'_m} h^{\frac{2+\beta}{2}}.$$

This concludes the proof of the upper bound in the case $0 \leq \beta < \frac{2}{3}$.

Remark 4.1. *It is worth mentioning that λ (recall (3.30)) and ℓ are related via*

$$\frac{h^{\frac{2+\beta}{4}}}{\ell} = C\lambda.$$

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